VISUAL COMPLEX ANALYSIS VIII

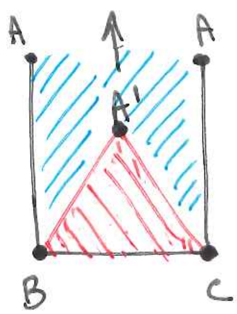
SCHWARTZ-CHRISTOFFEL MAPPING

SEPT. 24, 2012

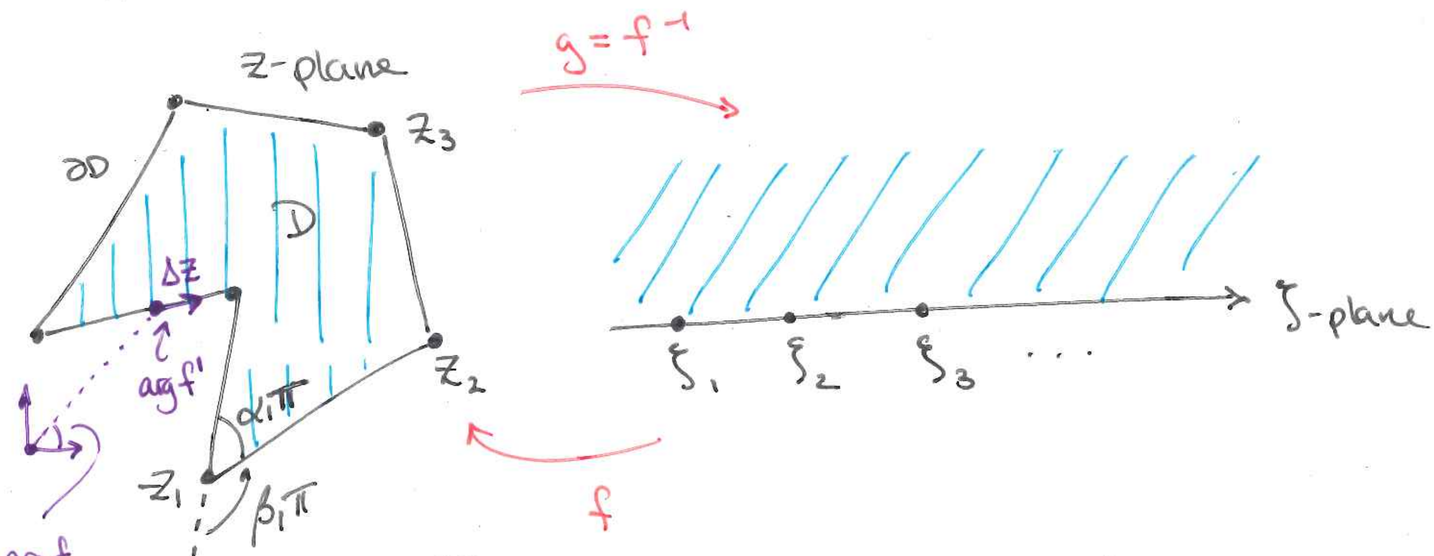
→ Provides an (implicit) formula for mapping <u>any</u> polygon to the half-plane.

→ Includes polygons with angles of 0° and 360°.

e.g.



(rectangle with one open side has vertices with angles 90°, 90° and 0° (seen as limiting case of $\Delta A'BC$))



Polygon D has interior angles $\alpha_1\pi, \alpha_2\pi, \ldots$ at vertices $z_1, z_2 \ldots z_n$

Define $\beta_j\pi = \pi - \alpha_j\pi$ as exterior angle $\Rightarrow \sum_{j=1}^{n} \beta_j = 2$

(note $\beta_j\pi < 0$ for oblique interior angles)

→ Note tangent to ∂D has direction $\arg[f'(\zeta)]$

Pf: $f(\zeta+\Delta\zeta) = f(\zeta) + f'(\zeta)(\Delta\zeta)$

real since $\partial D'$ is along axis

$\Rightarrow \arg[f(\zeta+\Delta\zeta)] = \arg[f(\zeta)] + \arg[f'(\zeta)]$

→ ∴ @ the point $\zeta = \zeta_j$,

$$\arg[f'(\zeta)]_{\zeta_j^-}^{\zeta_j^+} = \beta_j \pi \qquad (*)$$

∴ need a function s.t.

$$\arg[f_j'(\zeta)] = \begin{cases} 0 & \zeta > \zeta_j \\ -\beta_j \pi & \zeta < \zeta_j \end{cases}$$

→ Because then $f'(\zeta) = c \prod_{j=1}^{n} f_j(\zeta)$ works

exactly satisfying $(*)$.

→ This function is $f_j'(\zeta) = (\zeta - \zeta_j)^{-\beta_j}$

$\because \zeta > \zeta_j \Rightarrow \arg[f_j'(\zeta)] = 0$

$\zeta < \zeta_j \Rightarrow \arg[f_j'(\zeta)] = -\pi\beta_j$

$$\Rightarrow \quad \frac{df}{d\zeta} = C \prod_{j=1}^{n} (\zeta - \zeta_j)^{-\beta_j}$$

SCHWARZ-CHRISTOFFEL FORMULA

$$\Rightarrow \quad \boxed{f(\zeta) = C \int^{\zeta} \prod (t - \zeta_j)^{-\beta_j} \, dt + D}$$

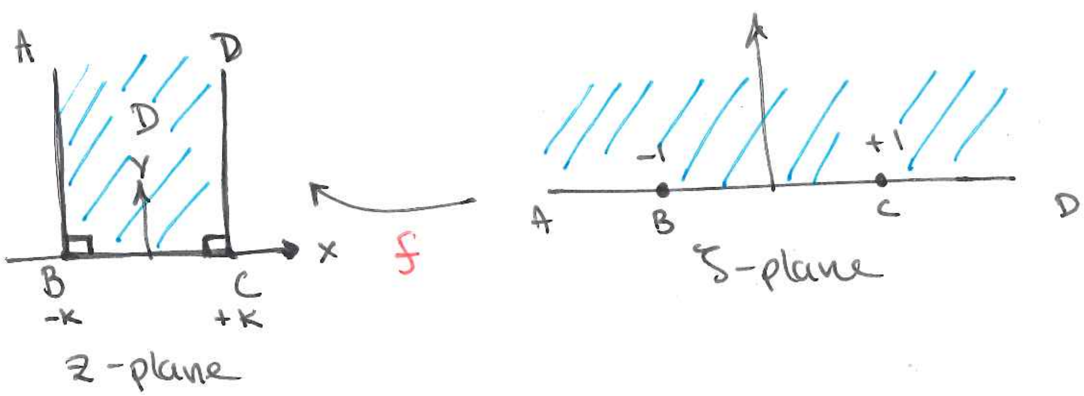
scaling + rotation

fixes location

→ note: • by RMT, you get to choose three image pts ζ_j

• Can choose $\zeta_j = \infty$, so that it is not involved in product, i.e. absorb into C.

Example:



choose 3 pts:

z_j	ζ_j	α_j	β_j
$-K$	-1	$\frac{1}{2}$	$\frac{1}{2}$
$+K$	$+1$	$\frac{1}{2}$	$\frac{1}{2}$
∞	∞	0	1

$$\Rightarrow f(\zeta) = c\int_0^{\zeta} (t+1)^{-\frac{1}{2}}(t-1)^{-\frac{1}{2}}\, dt + D.$$

choose for symmetry $\Rightarrow \zeta = 0 \mapsto z = 0$ if $D = 0$.

$$\Rightarrow f(\zeta) = \tilde{c}\int_0^{\zeta} \frac{1}{\sqrt{1-t^2}}\, dt$$

$$f(\zeta) = c\sin^{-1}(t)$$

c fixes side length $\Rightarrow$ need $f(+1) = +K$

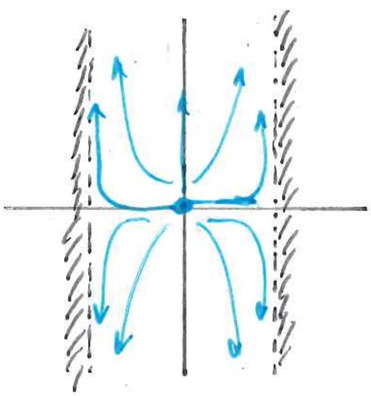
$$\Rightarrow c\left(\frac{\pi}{2}\right) = K \Rightarrow c = \frac{2K}{\pi}$$

$$\Longrightarrow f(\zeta) = \left(\frac{2K}{\pi}\right)\sin^{-1}\zeta$$

The map from z to ζ found by inverting:

$$\zeta = f^{-1}(z) = \sin\left(\frac{\pi z}{2K}\right)$$

Flow in an open rectangle? Place source @ $\zeta = 0$.



$$w(\zeta) = \frac{Q}{\pi}\log\zeta = \frac{Q}{\pi}\log\left[\sin\left(\frac{\pi z}{2K}\right)\right].$$