VISUAL COMPLEX ANALYSIS V

DIFFERENTIATION AND ANALYTIC FUNCTIONS

SEPT. 11, 2012

→ In 1D, $f'(x)$ is the slope of tangent to $f(x)$ at x. In 2D +, $\nabla f(x_1, \dots, x_n)$ allows us to compute rates of change of f along certain directions.

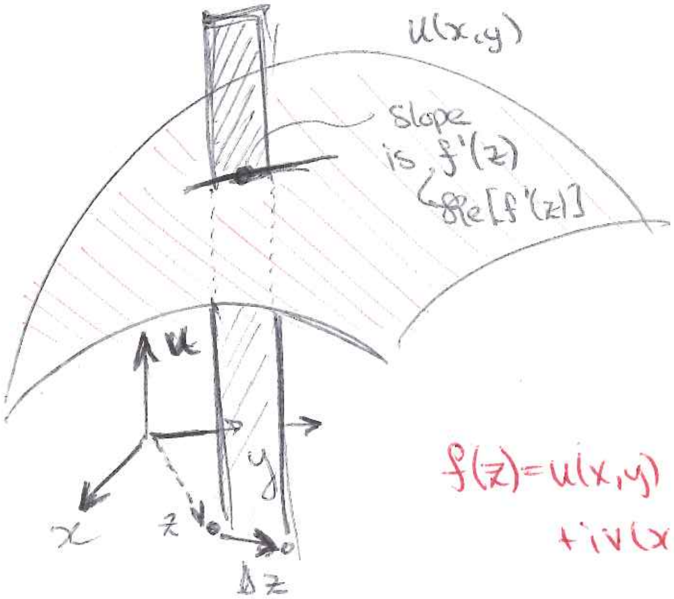
→ Suppose we stick with the same definition for complex functions. The derivative is defined as

$$f'(z) = \lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z} \qquad (*)$$

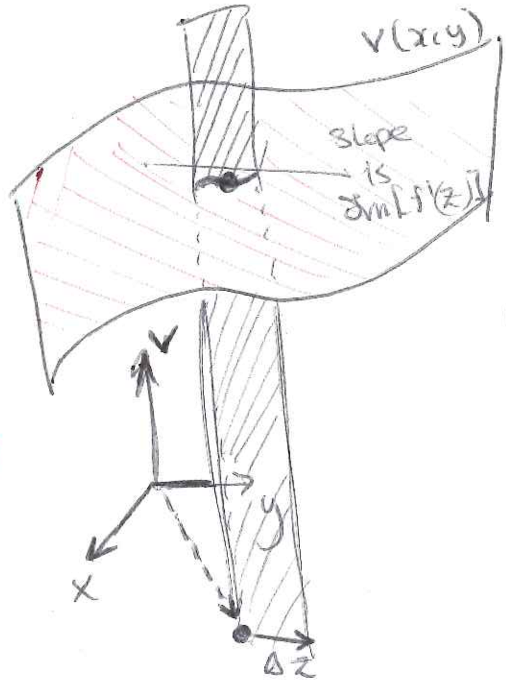
assuming this limit exists.

→ A function that is diff. on an open set $R \subseteq \mathbb{C}$ is said to be <u>analytic</u> or <u>holomorphic</u>.

→ Note that since $z = x+iy$, then $f(z)$ can be viewed as a function of x & y, $f(z) = u(x,y) + iv(x,y)$. So $(*)$ is saying that the limit must exist (and thus single valued) regardless of the direction for which $\Delta z \to 0$.

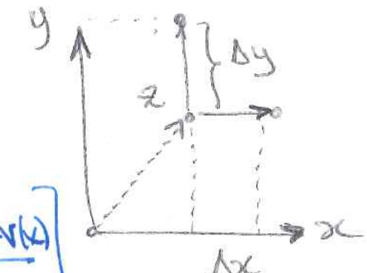


$f(z) = u(x,y) + i v(x,y)$



→ Moreover, this must be the same value for <u>every</u> possible $\Delta z \to 0$. ⇒ The existence of a diff. function is a <u>very</u> powerful statement!

→ In particular, the value of $f'(z)$ should be same regardless of $\Delta z = \Delta x \to 0$ or $\Delta z = i\Delta y \to 0$



$$f'(z) = \lim_{\Delta x \to 0} \frac{f(z+\Delta x) - f(z)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \left[\frac{u(x+\Delta x, y) - u(x,y)}{\Delta x}\right] + i\left[\frac{v(x+\Delta x, y) - v(x)}{\Delta x}\right]$$

$$= \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} \qquad (1)$$

→ Can you compute $f'(z)$ in the case $\Delta z = i\Delta y$?

$$f'(z) = \lim_{\Delta y \to 0} \left\{ \frac{u(x,y+\Delta y) - u(x,y)}{(i\Delta y)} + i\,\frac{v(x,y+\Delta y) - v(x,y)}{(i\Delta y)} \right\}$$

$$= \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y} \qquad (2)$$

These two expressions must be equal:

$$\Rightarrow \boxed{\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{\&} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.} \qquad (**)$$

CAUCHY-RIEMANN EQUATIONS

The CR eqns are a necessary condition for $f'(z)$ to exist ... but are they sufficient?

Assume that $u(x,y)$ and $v(x,y)$ satisfy $(**)$ and are continuously diff.

$$(3) \quad u(x+\Delta x, y+\Delta y) = u(x,y) + u_x\Delta x + u_y\Delta y + O(|\Delta z|^2)$$

$$(4) \quad v(x+\Delta x, y+\Delta y) = v(x,y) + v_x\Delta x + v_y\Delta y + O(|\Delta z|^2)$$

where $\Delta z = \sqrt{\Delta x^2 + \Delta y^2}$

Then:

$$\lim_{\Delta z \to 0} \left\{ \frac{u(x+\Delta x, y+\Delta y) - u(x,y)}{\Delta z} + i \frac{v(x+\Delta x, y+\Delta y) - v(x,y)}{\Delta z} \right\}$$

$$\doteq \lim_{\Delta z \to 0} \left\{ \frac{u_x \cdot \Delta x + \overset{-v_x}{u_y} \cdot \Delta y}{\Delta z} + i \frac{v_x \Delta x + \overset{=u_x}{v_y} \Delta y}{\Delta z} \right\}$$

$$= u_x \lim_{\Delta z \to 0} \left(\frac{\Delta x + i \Delta y}{\Delta z} \right) + i v_x \lim_{\Delta z \to 0} \left(\frac{\Delta x + i \Delta y}{\Delta z} \right) = \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x}$$

which we assume exists.

<u>THEOREM</u>: (CAUCHY-RIEMANN) The function $f(z) = u(x,y) + iv(x,y)$ is differentiable at $z = x+iy$ (and within a region of z) if and only if u_x, u_y, v_x, v_y are continuous and satisfy $u_x = v_y$ and $u_y = -v_x$.

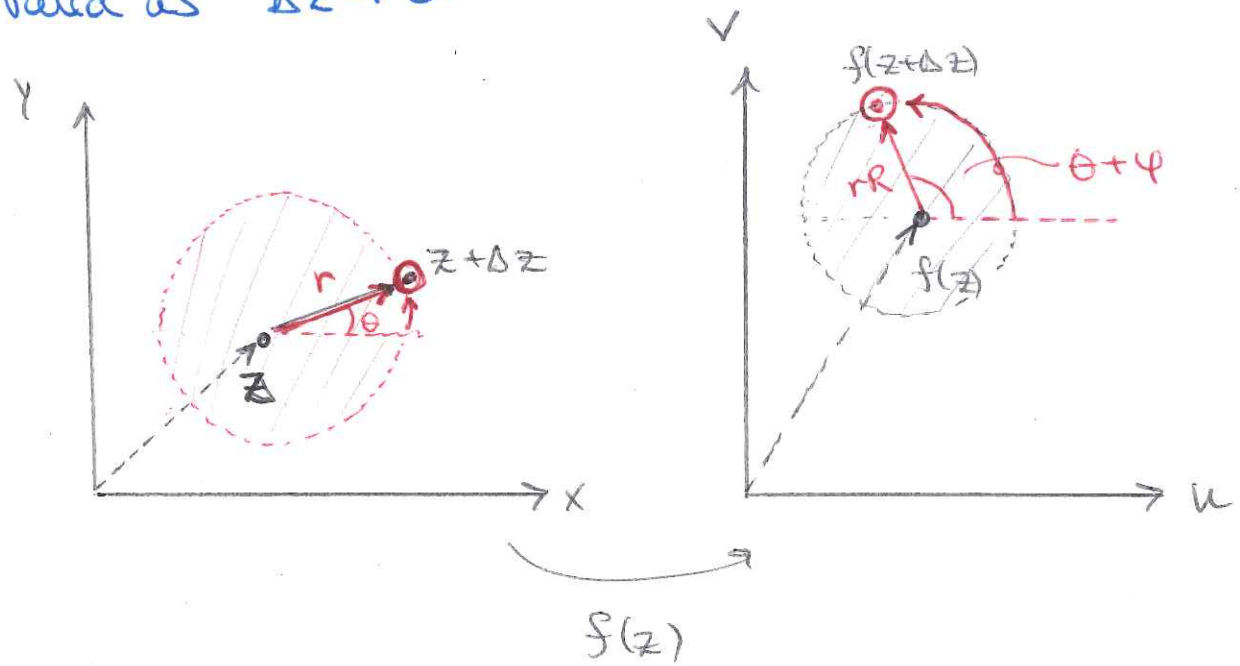
But how do we imagine differentiable functions? (In 1D, it is a question of having a smooth graph)

Moreover, what is a non-pathological $f(z)$ that is <u>not</u> differentiable?

We introduce (Needham's) the idea of the amplitwist. Re-write:

$$f(z+\Delta z) = f(z) + f'(z)\cdot \Delta z$$

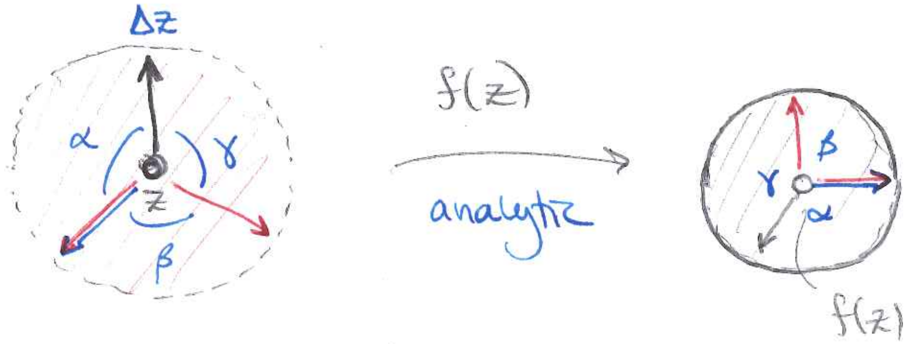
valid as $\Delta z \to 0$.

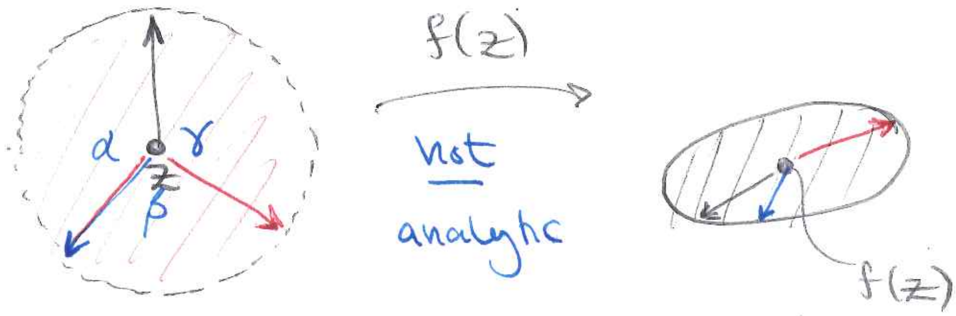


Suppose that $\Delta z = re^{i\theta}$ and $f'(z) = Re^{i\varphi}$.

Then we see that the image of the perturbation is $f(z+\Delta z) = f(z) + (rR)e^{i(\theta+\varphi)}$

So what is important is that there is only one R and φ and so all infinitessimal perturbations Δz are amplified and twisted by the same amount.





Instead of defining the notion of analytic functions as functions that possess derivatives, we can instead define:

Analytic functions are precisely those whose local effect is an amplitwist: all infinitessimal complex numbers are amplified and twisted by the same amount.

THEOREM: Assume $f(z)$ analytic and not constant on $D \subseteq \mathbb{C}$. For any $z \in D$ where $f'(z) \neq 0$, this mapping is conformal, i.e. it preserves angles between two arcs