SEPT. 5, 2012

VISUAL COMPLEX ANALYSIS 4

MULTIVALUED FUNCTIONS AND BRANCH CUTS (PART 2)

Review:

→ Review the idea of the function $f(z) = z^{\frac{1}{3}}$

- necessity of branch cut and choosing branches.
- Principal branch : $\theta = \pi$ cut and first branch,

$$(re^{i\theta})^{\frac{1}{3}} = r^{\frac{1}{3}} e^{\frac{i\theta}{3}}.$$

→ A <u>branch point</u> is a point for which, if $f(z)$ is discontinuous upon traversing the point in a small circle.

→ <u>An unconventional example:</u>

- I cut along $\theta = -\frac{\pi}{2}$. and $-\frac{\pi}{2} \leq \theta < 3\frac{\pi}{2}$
- I choose the second branch:

$$f(z) = f(re^{i\theta}) \equiv r^{\frac{1}{3}} e^{\frac{i\theta}{3}} e^{\frac{2\pi i}{3}}$$

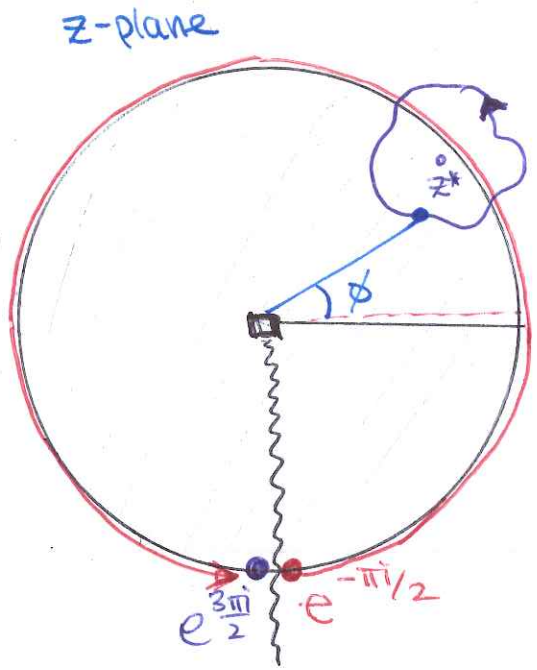
$$(re^{i\theta})^{\frac{1}{3}}$$

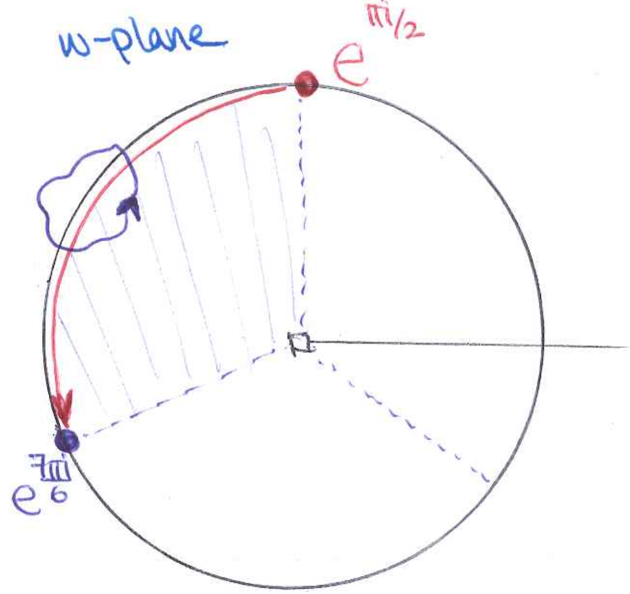
- Find values on either side of the cut:

If $r=1, \theta = -\frac{\pi}{2} \implies f(z) = e^{\frac{-\pi i}{6}} e^{\frac{2\pi i}{3}} = e^{\frac{3\pi i}{6}}$

If $r=1, \theta = 3\frac{\pi}{2} \implies f(z) = e^{\frac{\pi i}{2}} e^{\frac{2\pi i}{3}} = e^{\frac{7\pi i}{6}}$

notice discontinuity about branch cut

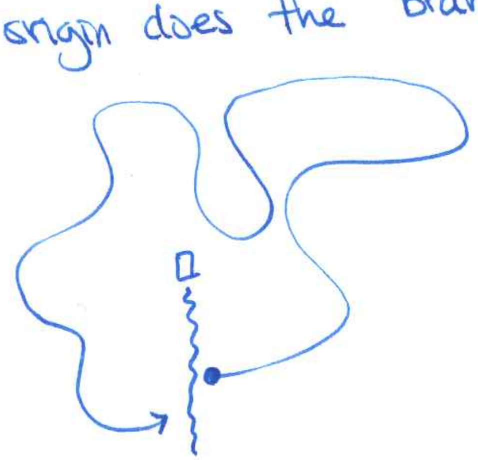




- Why is $z^* = re^{i\phi} \neq 0$ <u>not</u> a branch point?

  Consider a loop around z^*. The angle "rocks" back and forth around $\theta = \phi$ but returns to its original value. In the image plane, the image point does <u>not</u> go onto a different branch.

- Only by taking a full revolution around the origin does the branch change



$$\left[re^{i(\phi+\epsilon)}\right]^{\frac{1}{3}} = (r+r_\epsilon)^{\frac{1}{3}} e^{\frac{i(\phi+\epsilon)}{3}} e^{\frac{2\pi i}{3}}$$

the argument never picks up a full 2π around z^*

→ Multiple branch points:

The trick to dealing with a complex function involving multiple branch points is to separate the factors and examine the mapping locally near the critical points:

- Classic example:

$$f(z) = \sqrt{(z^2-1)} = \sqrt{(z-1)(z+1)}$$

- Can we write

$$f(z) = \underbrace{(z-1)^{\frac{1}{2}}}_{w_2}\ \underbrace{(z+1)^{\frac{1}{2}}}_{w_1} \stackrel{?}{=}$$

Expansion of powers only works if we clearly define how to interpret the function.

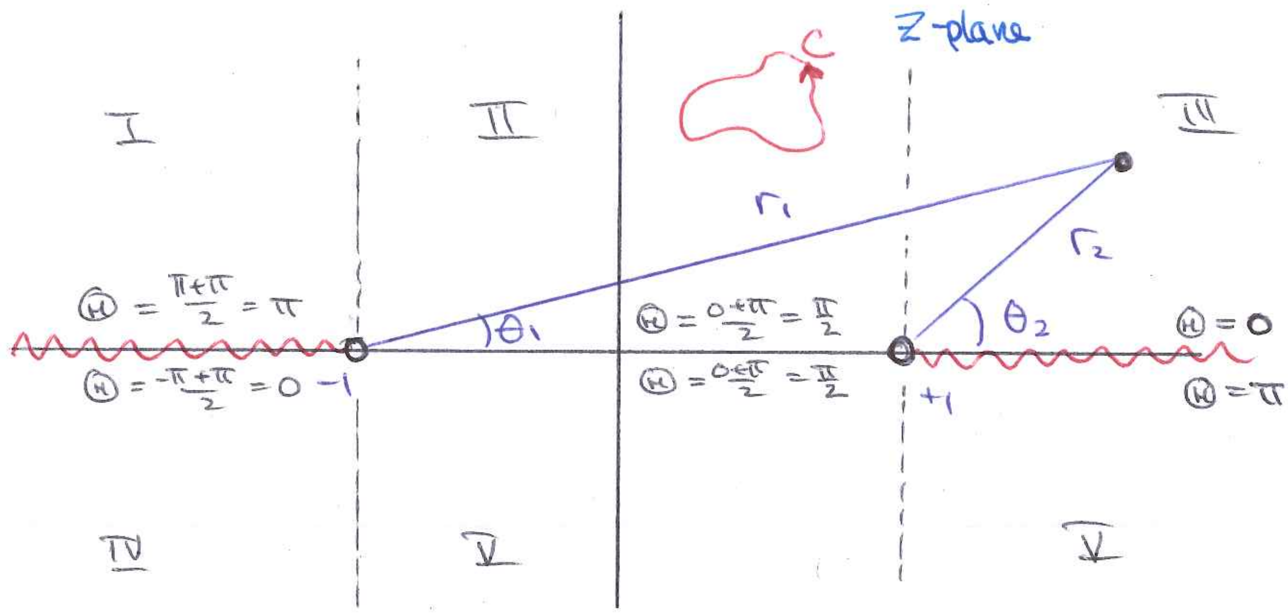
[note $\sqrt{(-1)(1)} = \sqrt{-1} = i$ but $\sqrt{(-1)(-1\cdot -1)} = i \cdot i \cdot i = -i$]

- Let us choose branch cuts like so 

$-\pi \le \theta_1 < \pi,\ 0 \le \theta_2 < 2\pi$

- For each factor, we pick the "first" branch of the $z^{1/2}$ function.

$$f(z) = w_1 \cdot w_2 = \left[r_1 e^{i\theta_1}\right]^{\frac{1}{2}} \left[r_2 e^{i\theta_2}\right]^{\frac{1}{2}} \equiv \left[r_1^{\frac{1}{2}} e^{\frac{i\theta_1}{2}}\right]\left[r_2^{\frac{1}{2}} e^{\frac{i\theta_2}{2}}\right]$$



- Once functions get complicated, it's difficult to illustrate the image plane.
- We want to check behaviour of $f(z)$ along x-axis. and notably in quadrants (I - VI)
- Let $\Theta = \frac{\theta_1 + \theta_2}{2}$ [The angle of $f(z)$].

  in the limit that $\mathrm{Im}(z) \to 0$.

  [fill in the picture].

- The only jump is across the branch cut.
- What about evaluating f along C?

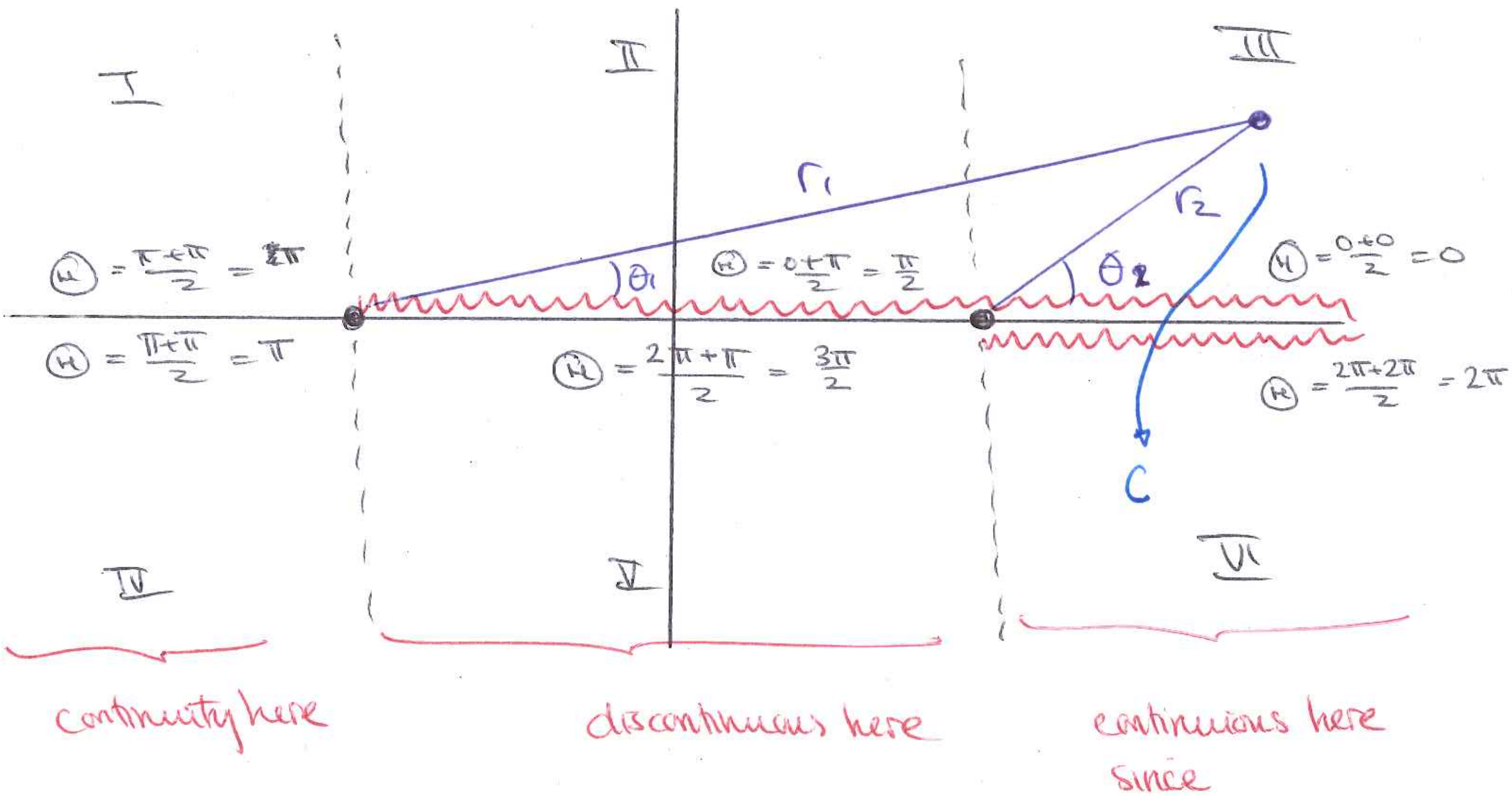
  No change in branch because both θ_1 and θ_2 oscillate but do not collect 2π.

Choose now cuts along $\theta_1 = 0$, $\theta_2 = 0$.

and the "first" branch:

$$f(z) = \left(r_1 e^{i\theta_1}\right)^{\frac{1}{2}} \left(r_2 e^{i\theta_2}\right)^{\frac{1}{2}}$$

$$= \left(r_1^{\frac{1}{2}} e^{\frac{i\theta_1}{2}}\right)\left(r_2^{\frac{1}{2}} e^{\frac{i\theta_2}{2}}\right)$$



$e^{i\Theta} = e^0 = 1$

$e^{2\pi} = 1$

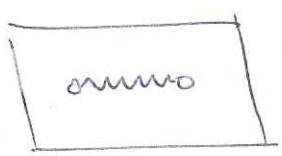
Because $\sqrt{(\ )}$ only has two branches

when we go from III to VI

along C, we negate once

when crossing branch cut due to $z=1$

then once again due to $z=-1$

The end picture is  but this is only because we have two square roots!

Future homework question: $f(z) = (z+i)^{\frac{1}{2}}(z-i)^{\frac{1}{3}}$.

→ Now cover the idea of a Riemann sheet, which unifies this whole subject.