Visual Complex Analysis II

BASICS & VISUALIZING FUNCTIONS

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1) BASICS

- A complex number, $z \in \mathbb{C}$ is an expression of the form

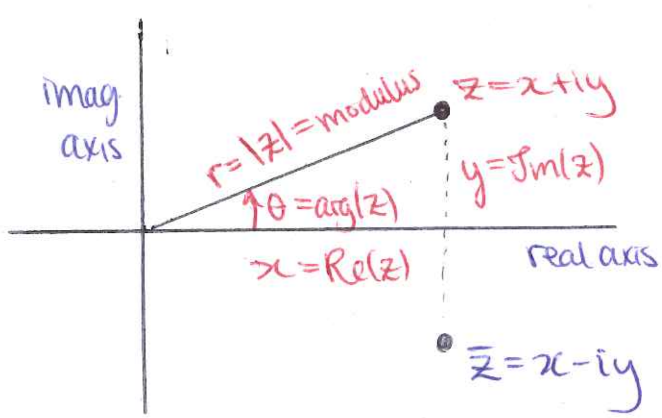
$$z = \underbrace{x}_{\Re e(z)} + i \underbrace{y}_{\Im m(z)}$$

where $i^2 = -1$. Geometrically, we represent this as a point in the (x,y) plane.

- Alternatively, we can also use polar coords.

$$z = r(\cos\theta + i\sin\theta)$$

- If $z_1 = z_2$ then $\Re e(z_1) = \Re e(z_2)$, $\Im m(z_1) = \Im m(z_2)$
- Addition: $(a+bi) + (c+di) = (a+c) + i(b+d)$
- Mult: $(a+bi)\cdot(c+di) = (ac-bd) + i(ad+bc)$



Name	Meaning	Notation
modulus	length of vector	$\|z\|, r$
argument	angle	$\theta, Arg(z)$
conjugate	reflection about x-axis	$\bar{z}$

Quick practice: Prove:

(i) $(1+i)^2 = 2i$ in two ways

(ii) $\overline{z_1 z_2} = \overline{z_1} \cdot \overline{z_2}$

(iii) $z\overline{z} = |z|^2$

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2) EULER'S FORMULA:

Euler's famous formula (~1740):

$$e^{i\theta} = \cos\theta + i\sin\theta.$$

Note that we can then write

$$z = x+iy = r(\cos\theta + i\sin\theta) = re^{i\theta}.$$

Multiplication will then be easy-peasy!

$$z_1 \cdot z_2 = (re^{i\theta})(Re^{i\phi}) = rRe^{i(\theta+\phi)}.$$

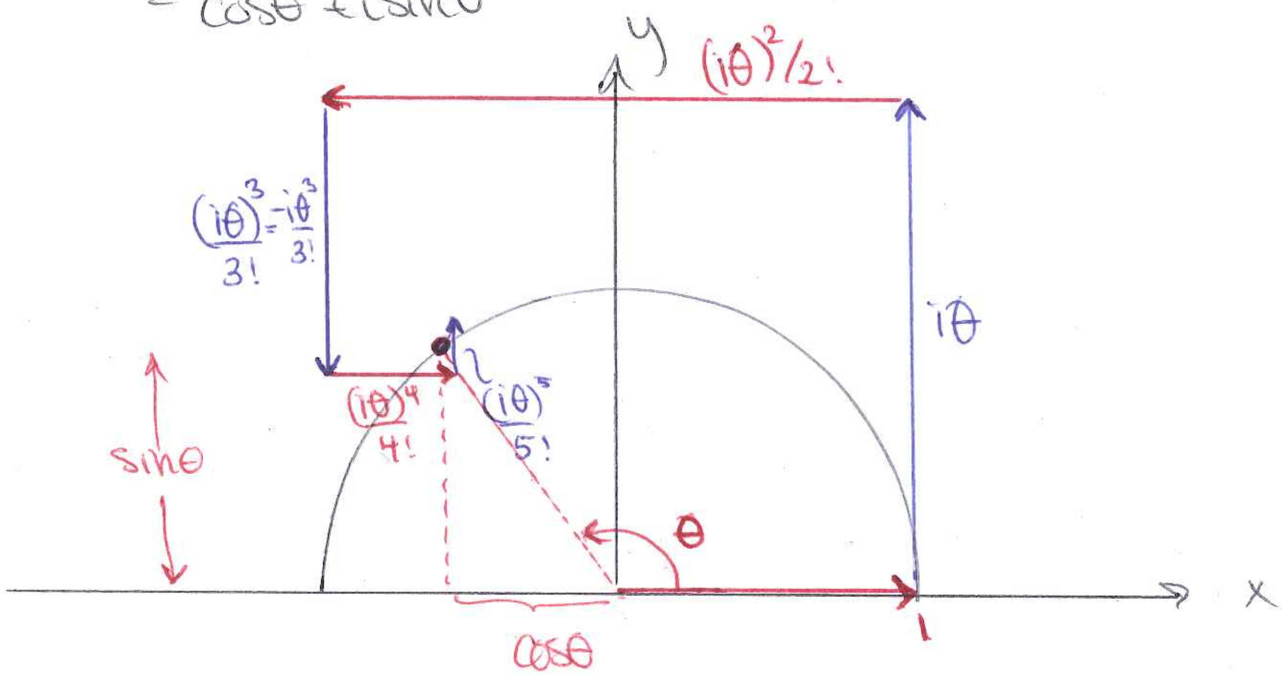
It's cheap to just define $e^{i\theta} = \cos\theta + i\sin\theta$. There are two basic ways to motivate this fact: Use the property that $\frac{d}{d\theta}(e^{i\theta}) = ie^{i\theta}$ [This is how e^x is defined]. Alternatively, we can assume the series for $e^x = 1 + x + \frac{x^2}{2!} + \cdots$ holds for complex values.

We shall write:

$$e^{i\theta} = 1 + (i\theta) + \frac{(i\theta)^2}{2} + \frac{(i\theta)^3}{3!} + \frac{(i\theta)^4}{4!} + \frac{(i\theta)^5}{5!}$$

$$= \underbrace{\left[1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} + \cdots\right]}_{\cos\theta} + i\underbrace{\left[\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} + \cdots\right]}_{\sin\theta}$$

$$= \cos\theta + i\sin\theta$$



Quick Practice:

Write $(1+i)(2+2\sqrt{3}\,i) = ?$ using exponentials.
[note $\tan\theta = \sqrt{3} \Rightarrow \theta = \frac{\pi}{3}$]

$$1+i = \sqrt{2}\,e^{\pi i/4}$$

$$2+2\sqrt{3}\,i = 4\cdot e^{\pi i/3}$$

$$\Rightarrow (1+i)(2+2\sqrt{3}\,i) = \sqrt{2}\cdot 4e^{i\left(\frac{\pi}{3}+\frac{\pi}{4}\right)}$$

3) COMPLEX VALUED FUNCTIONS.

- A complex function, $f : \mathbb{C} \to \mathbb{C}$ outputs a complex #.
  takes in a complex numb.

We have already encountered the basic complex functions:

$$\begin{cases} f(z) = iz \\ f(z) = z + (1+i) \\ f(z) = az + b, \quad a, b \in \mathbb{R}. \end{cases}$$

Difficulty with visualizing complex numbers is that it is 4D: if $w = u + iv = f(x+iy)$ need to imagine surface (x, y, u, v)

What do these look like? $\begin{cases} f(z) = z^3 \\ f(z) = \sin z \\ f(z) = (z+2)^2(z-1-2i)(z+i) \end{cases}$

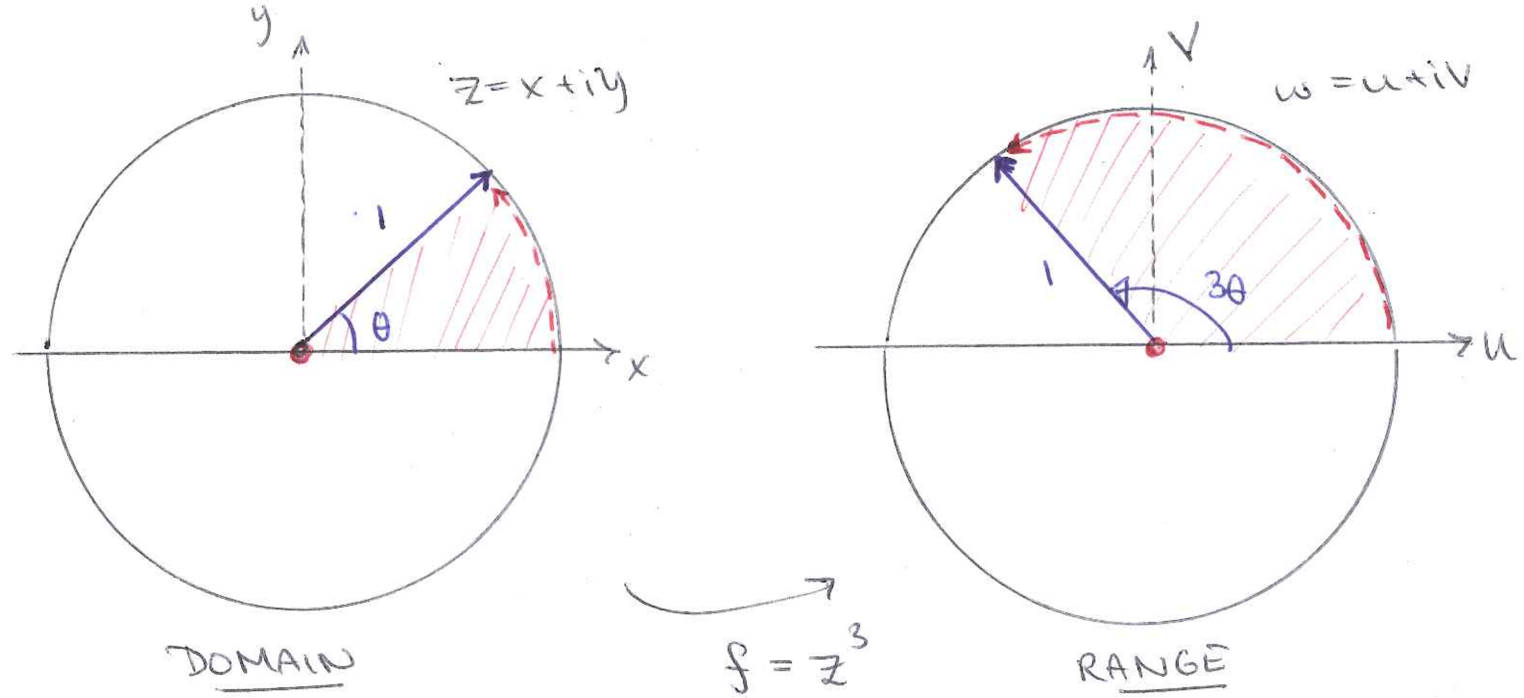
A) DOMAIN + RANGE PLOTS
(input) (output)

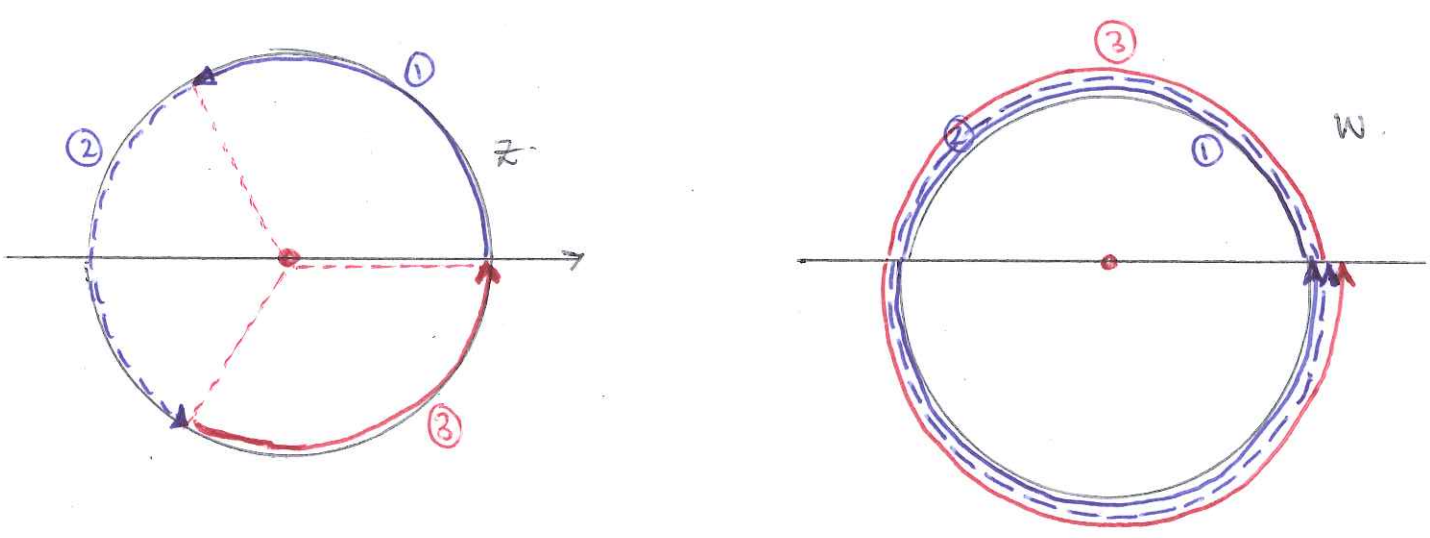
- We have already used these for studying mappings of Δs.

- Consider the $f(z) = z^3$ mapping
- Notice it is better to understand this map in polar coords:

$$\text{If } z = re^{i\theta} \Rightarrow f(z) = r^3 e^{3i\theta}$$

The mapping cubes the length of original vector and rotates the angle 3 times "as fast"





B) 3D SURFACES:

- Domain/range visuals are only useful for simple mappings that consist of only a few basic rotations/translations.

- Instead, we can plot a surface by choosing 3 pts from (x, y, u, v)

$$\text{e.g.} \begin{cases} (x, y, u) \\ (x, y, v) \\ (x, y, |u+iv|) \end{cases}$$

(note: the colour can represent the 4th dim.)

Example:

$$f(z) = \sin z = \frac{e^{iz} - e^{-iz}}{2i}$$

note along real axis, $f(z=x) = \sin x$.

" " imag axis, $f(z=iy) = \dfrac{e^{-y} - e^{y}}{2i} = \left(\dfrac{e^{+y}}{2} - \dfrac{e^{-y}}{2}\right)i$

[See Mathematica figure]

C) DOMAIN COLOURING.

D) The Riemann sphere.

Covered in slides + video.