350
Introduction to Differential Equations

This is your last problem set *(yay…?)* This one focuses on the techniques of separation of variables for solving the heat, wave, and Laplace's equation, and also the method of images for solving PDEs using the explicit form of the solution in terms of Green's functions.

Instructions:
Homework should be completed on loose-leaf paper stapled or bound together.
Write your name and class neatly at the top or on a separate title page.
Presentation matters and 5% of your mark will be on how you present your solutions.

Hand-in date: Wednesday April 9, 2012

1. **Building a root cellar**

We'd like to build an underground cellar to store our food. Let $u(x,t)$ be the temperature variation underneath your house from the annual mean, where $x > 0$ is the depth and t is time. The temperature at the Earth's surface is assumed to fluctuate according to

$$u(0,t) = a\cos(\omega t),$$

where a is constant and

$$\omega = \frac{2\pi}{3.15 \times 10^7 \text{ secs.}}$$

while at large depth, the temperature does not vary:

$$u(x,t) \to 0 \quad \text{as } x \to \infty.$$

The goal is to choose a depth, x, such that your cellar is mostly held at constant temperature throughout the year.

(a) By modeling the temperature variations using a simple 1D heat equation, solve for the temperature, $u(x,t)$.

(b) We define the optimal depth of the cellar as the smallest x in which the cellar's seasons are 6 months out of sync with the surface season. At that depth, thermal convection currents from the surface seasons would tend to move the cellar's temperatures closer to the annual mean, warming it during the winters, and chilling it during the summers.

For typical soils, $D \approx 0.002 \text{ cm}^2/\text{s}$. With this in mind, at what depth should you build your cellar. Make a sketch of the cellar temperature vs. the surface temperature as a function of the seasons.

2. **Separation of variables and the wave equation**

Consider the wave equation

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2},$$

where c is constant, and $u = u(x,t)$, defined on the finite interval $x \in [0, L]$.

(a) Solve the wave equation with the fixed (Dirichlet) boundary conditions:

$$u(0,t) = 0 = u(L,t)$$

by using separation of variables and expressing the solution as a Fourier series.

(b) Using $L = 2$, and beginning with the initial conditions of

$$u(x,0) = f(x) = e^{-(x-0.8)^2/0.01} \quad \text{and} \quad u_t(x,0) = 0.$$

What is the periodicity of the solution *in time*? That is, determine the smallest $t = t^*$ such that $u(x, t+t^*) = u(x,t)$. Using Matlab, numerically solve for the Fourier series coefficients using $N = 16$ modes, and create a single figure which plots the profile over 16 snapshots in time from $t = 0$ to $t = t^*$. You will want to use the command SUBPLOT.

(c) Physically interpret what you saw in the previous part by discussing: (i) how the initial profile $f(x)$ breaks into left and right traveling waves; (ii) what these waves do once they encounter the boundaries at $x = 0$ and $x = L$. It will be useful to use your intuition about how waves behave on for example, a 'jump rope'.

(d) We would like to be able to apply d'Alembert's wave formula to give us a different perspective of the solution to the wave equation. Let us instead consider the infinite interval $-\infty < x < \infty$ and create the odd, $2L$ periodic extension of $f(x)$, with the properties

$$\overline{f}(-x) = -\overline{f}(x) \quad \text{and} \quad \overline{f}(x + 2L) = f(x),$$

and $g(x) \equiv 0$. By using the method of characteristics, show that

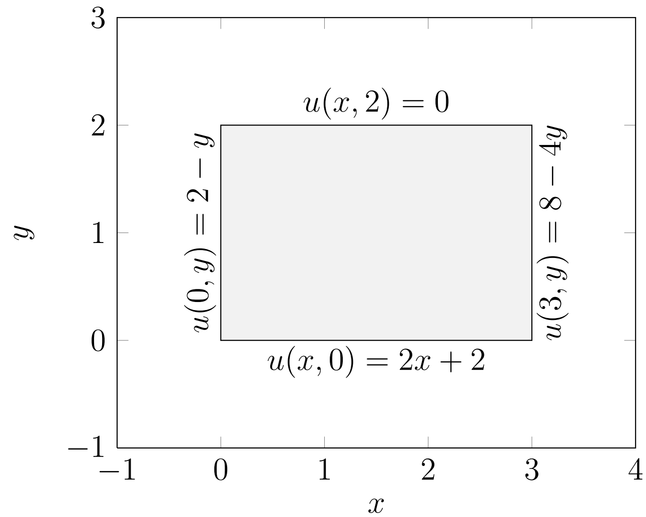
$$u(x,t) = \frac{\overline{f}(x - ct) + \overline{f}(x + ct)}{2}.$$

Finally, for a larger interval (*e.g.* $x \in [-10, 10]$), sketch different profiles at different times, showing how the initial displacement $\overline{f}(x)$ evolves over time.

Parts (c) and (d) both emphasize two related interpretations of the solutions to the wave equation. An observer who only sees the interval $[0, L]$ would say that the solution behaves like a wave (the Fourier interpretation). On the other hand, an observer who sees the entire extended domain $-\infty < x < \infty$ would say the solution behaves like particles, where wave packets collide and move on, unchanged in form (d'Alembert interpretation).

3. **Laplace's equation in a rectangle**

(a) By using separation of variables, find the solution of Laplace's equation within a rectangle $0 < x < 3$ and $0 < y < 2$ subject to the boundary conditions in the figure:



Explicitly determine all the necessary series coefficients. Use Matlab to plot the surface.

(b) Beginning from $(x, y) = (0, 0)$ and going around the rectangle in a counter-clockwise direction, make a graph of the boundary conditions as a function of the arc length ($s = [0, 10]$) along the rectangle. Does your series solution exactly equal the boundary conditions along the rectangle's edge? Why are why not?

(c) In fact, in cases where the boundary conditions of the rectangle are linear and continuous, there exists a simple solution. Try the ansatz:

$$u(x, y) = \alpha x + \beta y + \gamma xy + \delta,$$

as a possible solution for the PDE, and determine the constants $\alpha, \beta, \gamma, \delta$ using the correct boundary conditions. How does this solution differ from your Fourier series solution?

4. **GFs for a diffusion-absorption ODE** Consider the ODE given by

$$\mathcal{L}_x y(x) \equiv -\frac{d^2y}{dx^2} + q^2 y = f(x). \tag{1}$$

This equation would describe, for example, the diffusion of some substance produced at a rate $f(x)$, but also being degraded at a rate q^2y.

(a) By reposing in terms of a classically solvable problem, show that the free-space Green's Function, which satisfies

$$\begin{aligned} \mathcal{L}_x G(x,\xi) &= \delta(x-\xi) \quad \text{for } -\infty < x < \infty \\ G(x,\xi) &\to 0 \qquad\qquad \text{as } x \to \pm\infty \end{aligned}$$

is given by the expression

$$G(x,\xi) = \frac{e^{-q|x-\xi|}}{2q},$$

where we assume $q > 0$ without loss of generality.

(b) Assume that (1) is defined on the semi-infinite line $x \in [0,\infty)$ and with BCs

$$y(0) = 0 \quad \text{and} \quad y \to 0 \text{ as } x \to \infty$$

Using the method of images, write down the solution, $y(x)$, in terms of Green's Functions.

(c) Assume that (1) is defined on the semi-infinite line $x \in [0,\infty)$ and with BCs

$$y'(0) = 0 \quad \text{and} \quad y \to 0 \text{ as } x \to \infty$$

Using the method of images, write down the solution, $y(x)$, in terms of Green's Functions.

5. **Solving Poisson's Equation**

Consider Poisson's equation in $\mathbb{R}^2$:

$$\begin{aligned} \nabla^2 u(x,y) &= f(x,y) \qquad & (x,y) \in D \\ u(x,y) &= 0 \qquad & (x,y) \in \partial D. \end{aligned}$$

(a) The GF would then be defined according to

$$\begin{aligned} \nabla^2 G &= \delta(x-\xi)\delta(y-\eta) \qquad & (x,y) \in D \\ G(x,y;\xi,\eta) &= 0 \qquad & (x,y) \in \partial D. \end{aligned}$$

Show using Green's Identity that the solution can then be written as

$$u(x,y) = \iint\limits_D G(x,y;\xi,\eta) f(\xi,\eta)\, d\xi\, d\eta.$$

Explain how the formula needs to be adjusted if the boundary condition is instead inhomogeneous, with $u(x,y) = g(x,y)$ on ∂D.

(b) By using the fact that in polar coordinates, $\nabla^2 u(r,\theta) = \frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}$, show that the free-space Green's Function must be given by

$$G = \frac{1}{2\pi} \log\left|(x,y) - (\xi,\eta)\right|.$$

(c) Let D be quarter-half plane, with zero Dirichlet conditions on the axes:

$$D = \Big\{(x,y) | x \geq 0, y \geq 0\Big\}$$
$$u(x,0) = 0 \quad \text{and} \quad u(0,y) = 0$$

Find the associated Green's Function for this problem.

(d) Again, let D be quarter-half plane, but now, with Dirichlet/Neumann conditions

$$D = \Big\{(x,y) | x \geq 0, y \geq 0\Big\}$$
$$u_y(x,0) = 0 \quad \text{and} \quad u(0,y) = 0$$

Find the associated Green's Function for this problem.