350 *Introduction to Differential Equations*

This homework is essentially split into three parts: (i) finishing up the remnants of the series theory for ODEs, (ii) deriving certain PDEs from physical laws, and (iii) applying the method of characteristics to solve first-order PDEs.

Instructions:
Homework should be completed on loose-leaf paper stapled or bound together.
Write your name and class neatly at the top or on a separate title page.
Presentation matters and 5% of your mark will be on how you present your solutions.

Hand-in date: Wednesday April 18, 2012

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1. **Infinitely Differentiable $\Longleftrightarrow$ Analytic?**

   Functions that are analytic at $x = x_0$ are necessarily infinitely differentiable at that point (since the Taylor series is defined in terms of the function's derivatives). If a function, $f(x)$, is infinitely differentiable at a point $x = x_0$, is it necessarily analytic?

   Hint: Consider
   $$f(x) = \begin{cases} e^{-1/x}, & x > 0 \\ 0, & x \leq 0 \end{cases}$$

2. **A review of the four series expansion theorems**

In class, we covered three theorems for deriving the general solution of

$$y'' + p(x)y' + q(x)y = 0, \tag{1}$$

by using a series expansion around a point $x = x_0$: one theorem for ordinary points, and two 'Frobenius' theorems for regular singular points (one for distinct indicial values, $r = r_1, r_2$ not separated by an integer, and the second for repeated indicial values).

The missing *third Frobenius theorem* is as follows:

Let $x = x_0$ be a regular singular point of the ODE (1), and assume the two series

$$(x - x_0)p(x) = \sum_{n=0}^{\infty} p_n(x - x_n)^n \quad \text{and} \quad (x - x_0)^2 q(x) = \sum_{n=0}^{\infty} q_n(x - x_n)^n,$$

converge with radii of convergence, R_1 and R_2, respectively. Assume also that the roots of the indicial equation $r(r-1) + p_0 r + q_0 = 0$ are such that $r_1 > r_2$ and $r_1 - r_2 = N \in \mathbb{Z}^+$. The standard first solution is[1]

$$y_1 = |x - x_0|^r \sum_{n=0}^{\infty} a_n(r_1)(x - x_0)^n,$$

where we assume that the series coefficients are functions of r, or $a_n = a_n(r)$. Then the second solution is given by

$$y_2 = Ay_1 \log|x - x_0| + |x - x_0|^r \sum_{n=0}^{\infty} b_n(r_2)(x - x_0)^n,$$

where $b_n(r) = \frac{d}{dr}[(r - r_2)a_n(r)]$ and $A = \lim_{r \to r_2}(r - r_2)a_N(r)$. The radius of convergence of the resultant solutions is $R \geq \min\{R_1, R_2\}$.

By consulting your notes, write out a clear statement of the four series expansion theorems which address the solution of the ODE (1) in the case that $x = x_0$ is an ordinary or regular singular point.

[1] The natural log has been written log instead of ln as usual

3. **Deriving the independent solutions using series expansions**

   (a) Find two independent series solutions centered at $x_0 = 0$ (*i.e.* determine explicitly all coefficients) for the ODE

   $$x^2y'' + 5xy' + (4 - x)y = 0.$$

   (b) Find the *form* of each of two independent solutions around $x_0 = 0$ for the ODE

   $$xy'' - y = 0.$$

   Provide all details of the solutions up to the coefficients of the series terms. You need not determine the explicit form of the series coefficients, but do determine the underlying recurrence relation and show how the coefficients within each solution may then be computed.

4. **Derivation of the heat equation**

Assume we have a three-dimensional solid that is homogeneous, isotropic (meaning that the heat-conducting properties at any point are independent of direction), and continuous. Let $T(\mathbf{x}, t)$ be the temperature and $E(t)$ be the amount of heat energy, contained in a fixed volume, V. The point of this question is to derive the standard heat equation:

$$\frac{\partial T}{\partial t} = \left(\frac{\kappa}{\rho c}\right) \nabla^2 T. \tag{2}$$

(a) For most materials, the internal energy of the volume is directly proportional to the temperature, and this is written (where E is energy per volume):

$$E = \rho c T, \tag{3}$$

where ρ is density and c is the specific heat. Energy is measured in joules, $J = kg \cdot m^2/s^2$. If temperature is measured in Kelvins, K, look-up and state the SI units of density and specific heat, then verify that the units match-up in the above equation. Hereafter, the solid can be assumed to have constant density and specific heat.

(b) Define a heat flux vector, $\mathbf{q}(\mathbf{x}, t)$ and state its unit of measurement (in SI units). Fourier's heat law states that the heat flux is proportional to the negative of the temperature gradient, or

$$\mathbf{q} = -\kappa \nabla T, \tag{4}$$

where the constant, κ is the material's conductivity. What are the units of κ? Why does the negative sign make sense on physical grounds?

(c) The total internal energy of the volume is given by

$$\text{Internal energy} = \int_V E(\mathbf{x}, t)\, dV.$$

By applying conservation of energy to the solid, derive the heat equation (2), carefully stating all assumptions and steps. Verify that the units of the heat equation are consistent.

(d) In the above derivation, if instead the material's conductivity was a function of its position, $\kappa = \kappa(\mathbf{x})$, how would the heat equation change?

5. **Boundary conditions for the heat equation**

   (a) Write down the heat equation for the temperature, $T(x, t)$ of a long thin (one-dimensional rod), lying in $x \in [0, L]$.

   (b) If the rod is insulated at the end $x = 0$, show (using the physical justification of Q4) that the solution satisfies the boundary condition $\partial T/\partial x = 0$.

   (c) Show that the three-dimensional version of (b) leads to the analogous boundary condition $\partial T/\partial n = 0$, where n is the outward unit normal from the surface boundary.

   (d) A homogeneous, three-dimensional body occupying V is completely insulated. Its initial temperature is $T_0(\mathbf{x})$. What is the steady-state temperature that it reaches after a long time?

6. **Traffic flow and a speed-density relation**

Suppose that x measures distance along a road that is long enough so that, if we look at it from far enough away, the cars can be treated as a continuum with number density $\rho(x,t)$ (cars per kilometer) and speed $u(x,t)$. No cars join or leave the road.

(a) By posing an integral conservation law for the transport of cars, and then simplifying, derive the PDE:

$$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho u)}{\partial x} = 0, \tag{5}$$

which models the motion of the cars.

(b) We need an (experimental) relation between the speed of the cars and the density. First, it makes sense to assume that, for a uniform road, $u = U(\rho)$, so the speed of the drivers only depends on the local traffic. Next, let us assume that the maximum car speed on an empty road ($\rho = 0$) is $u = u_{\text{max}}$, while bumper-to-bumper traffic, with $u = 0$, has $\rho = \rho_{\text{max}}$. This suggests the law

$$u = u_{\text{max}} \left(1 - \frac{\rho}{\rho_{\text{max}}}\right). \tag{6}$$

Make a sketch of $u = U(\rho)$. We might think that, to minimize their journey times, each driver should drive close to their maximum speed. By examining the number of cars passing through an arbitrary point, x, explain why in light of the above velocity-density relation, this 'individually-oriented strategy' may not be the best.

(c) Consider now the PDE (5) with the speed-density relation (6), $u_{\text{max}} = 1$, $\rho_{\text{max}} = 1$, and an initial traffic profile of $\rho(0,x) = e^{-x^2}$. Solve the PDE using the method of characteristics. Be sure to derive the equation for the characteristics, $x = x(t)$, and the solution, written in implicit form.

(d) Use Matlab to plot out ≈ 100 evenly-spaced characteristics over the intervals $0 \le t \le 2$ and $-3 \le x \le 3$. Plot the surface of the (multi-valued) solutions ρ and u using your implicit formula. In addition to the output of your Matlab, hand-sketch a series of profiles of ρ and u at fixed times, in order to demonstrate the sequence.

(e) At what point $(t,x) = (t^*, x^*)$ does the solution first develop a shock? If the position of the shock is $x = s(t)$, at what speed does the shock propagate?

7. **Traffic at a red light**

Consider traffic on a long road which satisfies the PDE (5) and velocity-density relation (6) with $u_{\text{max}} = 1$ and $\rho_{\text{max}} = 1$. The cars are originally moving with constant density and speed, ρ_0, $u_0 \in [0, 1]$. Thus, the cars see the initial condition

$$\rho(0, x) = \rho_0.$$

At time $t = 0$, a red light is switched on at $x = 0$, so that for all subsequent time,

$$u(t, 0) = 0, \qquad \text{for } t > 0.$$

(a) Consider the cars that did not make the red light, that is $x < 0$. Apply the method of characteristics to solve the PDE; you will need to consider both sets of characteristics from the $(0, x)$-axis, and the $(t, 0)$-axis. Depending on whether there is light (ρ_0 is small) or heavy (ρ_0 is larger) traffic, sketch the characteristics in the xt-plane. If a shock occurs along $x = s(t)$, apply the Rankine-Hugoniot conditions to derive the location of the shock. What does the shock physically correspond to?

(b) Perform the same analysis as above but for the cars that made it past the red light, $x > 0$.

(c) Sketch profiles $\rho(t, x)$ as a function of x and for a series of snapshots in time in order to illustrate the general movement of the traffic. Does this correspond to what your intuition tells you for the traffic at a red light? Physically, what do the shocks derived in the previous two parts correspond to?