350 *Introduction to Differential Equations*

This homework continues our investigations of qualitative methods for the study of systems of ordinary differential equations. Afterwards, you will tackle some issues related to existence and uniqueness. The last topic will address series solutions of ordinary differential equations.

Instructions:
Homework should be completed on loose-leaf paper stapled or bound together.
Write your name and class neatly at the top or on a separate title page.
Presentation matters and 5% of your mark will be on how you present your solutions.

Hand-in date: Wednesday Mar. 28, 2012

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1. Consider the autonomous system

$$\frac{dx}{dt} = x, \qquad \frac{dy}{dt} = -2y + x^3.$$

   (a) Show that the critical point $(0, 0)$ is a saddle point.

   (b) Sketch the trajectories for the corresponding linear system and show that the trajectory for which $x \to 0$, $y \to 0$ as $t \to \infty$ is given by $x = 0$.

   (c) Determine the trajectories for the nonlinear system for $x \neq 0$ by integrating the equation for dy/dx. Show that the trajectory corresponding to $x = 0$ for the linear system is unaltered, but that for the one corresponding to $y = 0$ is $y = x^3/5$. Sketch several trajectories for the nonlinear system.

2. The system of differential equations

$$\frac{dx}{dt} = x(1.5 - 0.5x - y) \tag{1}$$
$$\frac{dy}{dt} = y(2 - y - 1.125x), \tag{2}$$

can be interpreted as describing the interaction of two species with populations x and y.

(a) Draw a direction field and describe how solutions seem to behave.

Hint: You've done direction fields for single differential equations in the $(t, x(t))$-plane. For this problem, you can do a direction field in the (x, y) plane where at each point, a small arrow is used to indicate the trajectory of a solution for increasing t.

(b) Find the critical points

(c) For each critical point, find the corresponding linear system. Find the eigenvalues and eigenvectors of the linear system; classify the type of each critical point, and determine whether it is asymptotically stable, stable, or unstable.

(d) Sketch the trajectories in the neighborhood of each critical point

(e) Determine the limiting behaviour of x and y as $t \to \infty$ and interpret the results in terms of the populations of the two species.

3. **Lipschitz and Picard iterates**

   (a) Show that $f(x, y) = \sqrt{y+1}$ is Lipschitz in y for $-1 \leq x \leq 1$ and $y > 0$. Find an appropriate Lipschitz constant.

   (b) Show that all the successive approximations for the problem

   $$y' = y^2, \qquad y(0) = 1,$$

   exist for all real x.

   (c) Find a solution of the initial value problem in the previous part. On what interval does it exist?

   (d) Assuming there is just one solution of the problem in (b), indicate why the successive approximations in (b) can not converge to a solution for all real x.

4. **Existence and Uniqueness**

In class, we proved Picard's Theorem by beginning with a rectangle,

$$R = \{(x, y) : |x - x_0| \leq h, |y - y_0| \leq k\},$$

such that

$$Mh \leq k \qquad \text{with } M = \max_{(x,y)\in R} |f(x, y)|.$$

This question is simply here to help you redo the proof for yourself on a slightly different variant of Picard's Theorem.

(a) This time, begin with an arbitrary rectangle

$$R = \{(x, y) : |x - x_0| \leq a, |y - y_0| \leq b\}.$$

where $a, b > 0$. For the IVP,

$$y' = f(x, y) \qquad \text{with } y(x_0) = y_0,$$

assume that f satisfies a Lipschitz condition of the second argument with constant K in R. Show that we can guarantee that there exists unique solution on the interval

$$I : \qquad |x - x_0| \leq \alpha = \min\{a, b/M\},$$

where $M = \max_{(x,y)\in R} |f(x, y)|$.

(b) Can you explain, intuitively, the difference between the two versions of Picard's Theorem?

5. **Existence and Uniqueness**

Using Picard's theorem for the IVP $y' = f(x, y)$ with given initial value, briefly discuss what conclusions may be drawn as to the existence and uniqueness of solutions.

(a)
$$f(x, y) = e^{xy} \qquad \text{with } y(0) = 1.$$

(b)
$$f(x, y) = |y| + |x| \qquad \text{with } y(0) = 0.$$

(c)
$$f(x, y) = \begin{cases} y^2 - x & \text{if } x \geq 0 \\ y - x & \text{if } x < 0 \end{cases}$$

6. **Classification of points**

   For the following ODEs, determine if the solution $y(x)$ is analytic at the point $x_0 = 0$ and if so state the minimum radius of convergence

   (a) $y'' + \frac{x}{x^2+2}y' + \ln(x+2)y = 0$

   (b) $xy'' + (\cos x)y' + e^x y = 0$

   (c) $xy'' + (\sin x)y' + (e^x - 1)y = 0$

   (d) $(x^2 - 4)y'' + (\tan x)y' + \sqrt{x^2 + 3}y = 0$

7. **Series solutions of ordinary differential equations**

   Find two linearly independent solutions of the differential equation

   $$y'' + xy' - 2y = 0,$$

   by performing a series expansion about the point $x_0 = 0$.

8. **Bessel's equation**

The differential equation,

$$z^2w''(z) + zw'(z) + (\lambda^2 z^2 - \nu^2)w(z) = 0,$$

where $\nu \geq 0$ and λ are constants, is called Bessel's equation of order ν with parameter λ. When $\lambda = 1$, then the equation is simply called Bessel's equation of order ν.

(a) Assuming that $\lambda \neq 0$, make the change of variables $x = \lambda z$ and $w(z) = y(x)$ to transform the equation to

$$x^2y'' + xy' + (x^2 - \nu^2)y(x) = 0,$$

i.e. Bessel's equation of order ν.

(b) Consider now $x > 0$ and search for a series expansion near $x_0 = 0$. Place the equation in standard form, using

$$y'' + p(x)y' + q(x) = 0,$$

and classify the point $x_0 = 0$ as ordinary, regular singular, or irregular singular. Solve the indicial equation for r_1 and r_2.

(c) Assuming that $r_1 - r_2$ is *not* an integer, show that two independent solutions can be derived as

$$y_1 = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\Gamma(\nu + k + 1)} \left(\frac{x}{2}\right)^{2k+\nu} \equiv J_\nu(x).$$

called the Bessel function of the first kind of order ν, and also

$$y_2(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\Gamma(k - \nu + 1)} \left(\frac{x}{2}\right)^{2k-\nu} \equiv J_{-\nu}(x).$$

Hint: The Gamma function, Γ, has the important property that $p\Gamma(p) = \Gamma(p+1)$, which is true for all p real except when p is zero or one of the negative integers. You can use this property of the Gamma function to simplify the coefficients of the series.

9. **Numerical solution of the Bessel equation**

In this question, you will investigate the Bessel function, $J_0(x)$, you derived in the previous question. This function is the solution of the IVP

$$x^2y'' + xy' + x^2y = 0, \qquad \text{with } y(0) = 1 \text{ and } y'(0) = 0,$$

over the interval $x \in I = [0, 50]$.

(a) Re-write the second order problem as a standard first order system. Can you explain why there is some difficulty in implementing a numerical solver which begins at $x = 0$? For the remainder of this problem, you may approximate the numerical solution of the IVP by instead using the initial conditions $y(\epsilon) = 1$ and $y'(\epsilon) = 0$ for $\epsilon = 10^{-12}$, and instead solve the problem over $I^* = [\epsilon, 50]$.

(b) Write your own Runge-Kutta (RK4) code to solve the differential equation over I^* using $N = 2^k$ equally spaced mesh points for $k = 6, 7, 8, 9, 10$.

(c) Investigate the built-in Matlab function BESSELJ. Use this function to compute $J_0(x)$, and make a plot of the exact solution overlaid with the numerical approximations you computed in the last part.

(d) Create a plot of the error (infinity norm) versus the number of mesh points from the last two parts. As we stated in class, the RK4 method should typically provide approximations with an error of $O(h^4)$, and yet, the error of your computations should be much worse. Can you make a conjecture about why this is?

(e) Use Matlab's built-in ODE45 function to compute the solution of the differential equation over I^*. Set both the absolute tolerance and the relative tolerance to 10^{-6}. How much mesh points did the solution use? Can you comment on the effectiveness of the ODE45 function versus your hand-coded RK4 routine?