350 *Introduction to Differential Equations*

This homework is begins with our classes on inhomogeneous problems, and ends with phase plane theory of mostly linear systems of differential equations. The numerical problem asks you to solve a differential equation using one of the techniques we learned in class.

Instructions:
Homework should be completed on loose-leaf paper stapled or bound together.
Write your name and class neatly at the top or on a separate title page.
Presentation matters and 5% of your mark will be on how you present your solutions.

Hand-in date: Wednesday Mar. 14, 2012

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1. **Inhomogeneous problems**
   (a) Find the general solution of $2y'' + 3y' + y = t^2 + 3\sin t$
   (b) Find the general solution of $y'' - 2y' - 3y = 3te^{2t}$
   (c) By assuming continuity of the solution and its first derivative, solve the following initial value problem:
   $$y'' + y = \begin{cases} t & 0 \leq t \leq \pi \\ \pi e^{\pi - t} & t > \pi \end{cases},$$
   where $y(0) = 0$ and $y'(0) = 1$. Plot a graph of the non homogeneous term and the solution as a function of time.
   (d) Consider the differential equation $ay'' + by' + cy = g(t)$, where a, b, c are positive. If $Y_1(t)$ and $Y_2(t)$ are both particular solutions of this equation, show that $Y_1(t) - Y_2(t) \to 0$ as $t \to \infty$. Does this still hold if $b = 0$?

2. **Resonance**

Early in the course, it was mentioned that systems involving a mass attached to a spring can often be described by an equation of the form

$$m\ddot{y} + \gamma\dot{y} + ky = F_0 \cos(wt),$$

where $y = y(t)$ is the displacement as a function of time, $m > 0$ is the mass of the object, $k > 0$ is the spring constant, and $\gamma \geq 0$ is a friction or damping parameter. Also, $F_0 \cos(wt)$, with F_0, w_0 both real constants, is a periodic external force which is applied to the mass.

Consider the forced but undamped system described by the IVP:

$$\ddot{y} + y = 3\cos(wt), \qquad \text{with} \quad y(0) = 0, \dot{y}(0) = 0.$$

(a) Find the solution for $w \neq 1$.

(b) Use computer software (*e.g.* Matlab) to plot solutions (as a function of time) for values of w ranging from $w = 0$ to $w = 1$. In your solution sheets, make hand sketches of some of these solutions.

(c) Prove the following statement: in the limit that $w \to 1$, the solution is a 'rapidly' oscillating sinusoidal of frequency (wavenumber)

$$\frac{(1+w)t}{2},$$

which is contained in an *envelope*[1] given by a slowly varying sinusoidal amplitude of

$$\pm\frac{6}{(1-w^2)} \sin\left(\frac{(1-w)t}{2}\right).$$

(d) Derive the solution of the IVP when $w = 1$. Can you explain what occurs to the envelope you derived in the previous part? What has this got to do with the phenomenon of *resonance*?

[1]By "envelope", we mean the two curves, $v_1(x)$ and $v_2(x)$ for which $v_1(x) \leq y \leq v_2(x)$, and v_1 and v_2 are the greatest and lowest (respectively) such curves

3. **Variation of parameters**

   (a) Find the general solution of $\ddot{y} + 4\dot{y} + 4y = t^{-2}e^{-2t}$ for $t > 0$

   (b) Find the general solution of $t\ddot{y} - (1+t)\dot{y} + y = t^2e^{2t}$ for $t > 0$, given that $y_1(t) = 1 + t$ and $y_2(t) = e^t$ are two solutions of the homogeneous equation.

4. **A behavioral question**

Consider the equation

$$\mathcal{L}[y] = \ddot{y} + a_1\dot{y} + a_2 y = b(x), \tag{3}$$

where a_1 and a_2 are constants and b is a continuous function on $0 \leq x < \infty$. Suppose the roots, r_1, r_2, of the characteristic equation are distinct, and $\Re(r_1) < 0$ and $\Re(r_2) < 0$.

(a) Suppose b is bounded on $0 \leq x < \infty$. Show that every solution of (3) is bounded on $0 \leq x < \infty$.

(b) If $b \to 0$ as $x \to \infty$, show every solution of $\mathcal{L}[y] = b(x)$ tends to zero as $x \to \infty$.

5. **Conversion to systems of differential equations**

(a) Change the following initial value problem into a first order matrix equation for some dependent variable, $\mathbf{x} = \mathbf{x}(t)$ (which you should define):

$$\ddot{x} = \dot{x} + \dot{y} - z + t \tag{4}$$
$$\ddot{y} = tx + \dot{y} - 2y + t^2 + 1 \tag{5}$$
$$\dot{z} = x - y + \dot{y} + z, \tag{6}$$

where $x(1) = 1$, $\dot{x}(1) = 15$, $y(1) = 0$, $\dot{y}(1) = -7$, and $z(1) = 4$.

(b) Show that if a_{11}, a_{12}, a_{21}, and a_{22}, are constants, with a_{12} and a_{21} not both zero, and if the functions g_1 and g_2 are differentiable, then the initial value problem

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} g_1(t) \\ g_2(t) \end{bmatrix},$$

with $\mathbf{x}(0) = [x_1^0 \, x_2^0]^T$ can be transformed into an initial value problem for a single second order equation. Can the same procedure be carried out if $a_{11}, \ldots, a_{22}$ are functions of t?

6. **Repeated eigenvalues**

Consider the problem $\dot{\mathbf{x}} = A\mathbf{x}$ where

$$A = \begin{bmatrix} 1 & -1 \\ 1 & 3 \end{bmatrix}$$

(a) Show that there is a repeated eigenvalue, r, but only a single independent eigenvector, ξ. Based on this, derive the single independent solution,

$$\mathbf{x}^{(1)} = \xi e^{rt}.$$

(b) Based on what was done for the case of scalar-function second-order ODEs, it makes sense to search for the second independent solution by trying

$$\mathbf{x}^{(2)} = \xi t e^{rt}.$$

Show that this ansatz will not work.

(c) The correct form is,

$$\mathbf{x}^{(2)} = \xi t e^{rt} + \eta e^{rt},$$

where η is called the **generalized eigenvector corresponding to** r. Solve for η and thus derive the general solution of the problem.

(d) By carefully arguing about the behaviour of the solutions as $t \to \pm\infty$, sketch the trajectories in the phase plane. The fixed point, $\mathbf{x} = 0$, is called an **improper node** in this case.

7. **Systems of equations**

(a) Consider the linear system $\dot{\mathbf{x}} = A\mathbf{x}$, where A is constant 2×2 matrix with real entries. By considering the two eigenvalues, r_1 and r_2, of A, fill in the following table. In the last column, make a small phase plane sketch of the critical point (*e.g.* 2 cm by 2 cm sketch)

Eigenvalues	Type of critical point	Stability	Sketch
$r_1 > r_2 > 0$			
$r_1 < r_2 < 0$			
$r_2 < 0 < r_1$			
$r_1 = r_2 > 0$			
$r_1 = r_2 < 0$			
$r_{1,2} = \lambda \pm i\mu$, $\lambda > 0$			
$r_{1,2} = \lambda \pm i\mu$, $\lambda < 0$			
$r = \pm i\mu$			

Possible critical points: center, spiral, node, improper node, proper node, saddle point.

Possible stability types: unstable, stable, asymptotically stable

(b) Now suppose

$$\frac{d\mathbf{x}}{dt} = \begin{bmatrix} f(x_1, x_2) \\ g(x_1, x_2) \end{bmatrix}$$

where f and g are continuous and differentiable functions over the domain of interest. Assuming that there is a fixed point at $\mathbf{x} = \mathbf{x}_0$, explain how we can approximate the nonlinear system of equations by an equation of the linear equation $\dot{\mathbf{x}} = A\mathbf{x}$. Finally, create a version of the above table, which is applied to eigenvalues of the linear approximation.

8. **Qualitative analysis**

Complete the following questions. In addition to finding the critical points, for each subquestion and each critical point, you should: (i) write down the linearised system and the eigenvalues, (ii) fill in the blacked-out squares, which correspond to types of critical points.

In Problems 1 through 8, find the critical point or points of the given autonomous system, and thereby match each system with its phase portrait among Figs. 6.1.11 through 6.1.18.

1. $\frac{dx}{dt} = 2x - y, \quad \frac{dy}{dt} = x - 3y$
2. $\frac{dx}{dt} = x - y, \quad \frac{dy}{dt} = x + 3y - 4$
3. $\frac{dx}{dt} = x - 2y + 3, \quad \frac{dy}{dt} = x - y + 2$
4. $\frac{dx}{dt} = 2x - 2y - 4, \quad \frac{dy}{dt} = x + 4y + 3$
5. $\frac{dx}{dt} = 1 - y^2, \quad \frac{dy}{dt} = x + 2y$

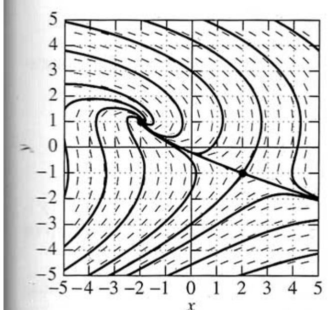

FIGURE 6.1.11. [redacted] point (−2, 1) and [redacted] point (2, −1).

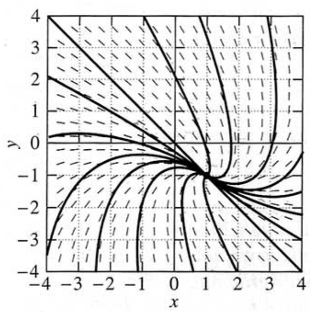

FIGURE 6.1.12. [redacted] point (1, −1).

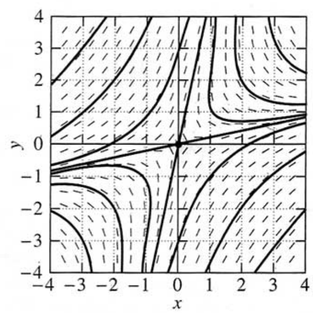

FIGURE 6.1.13. [redacted] point (0, 0).

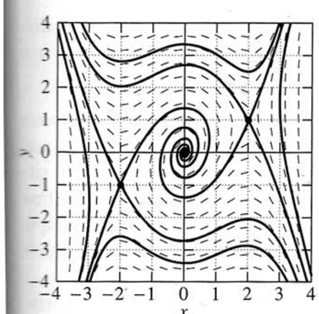

FIGURE 6.1.14. [redacted] point (0, 0); [redacted] points (−2, −1) and (2, 1).

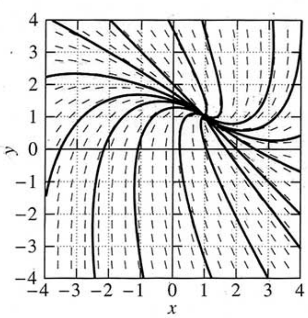

FIGURE 6.1.15. [redacted] (1, 1).

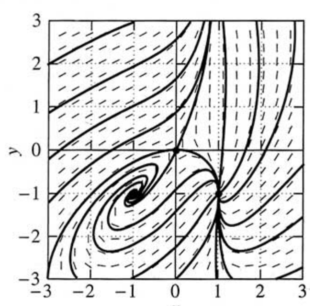

FIGURE 6.1.16. [redacted] point (−1, −1), [redacted] point (0, 0), and [redacted] (1, −1).

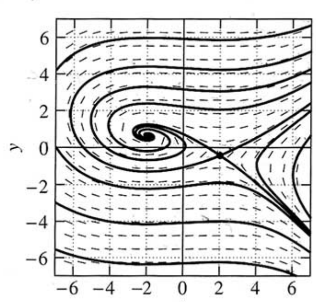

FIGURE 6.1.17. [redacted] point $\left(-2, \frac{2}{3}\right)$ and [redacted] point $\left(2, -\frac{2}{5}\right)$.

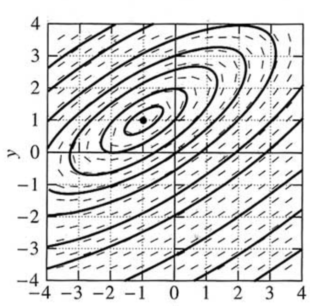

FIGURE 6.1.18. Stable [redacted] (−1, 1).

9. **A numerical approximation**

Consider the initial value problem:

$$y' = 5y - 6e^{-x}, \qquad \text{with } y(0) = 1 \text{ and } x > 0.$$

(a) Solve the IVP exactly.

(b) Write a Matlab code which implements the improved Euler method to solve for the numerical approximation for step sizes of $h = 10^{-1}, 10^{-2}, 10^{-3}, 10^{-4}, 10^{-5}$ on the interval $x \in [0, 5]$. Make a plot of the solutions overlaid on top of the exact solution. Re-scale the y limits of your plot to $y \in [-3, 1]$.

(c) Can you explain why the numerical solution spectacularly fails to converge to the exact solution?