350 *Introduction to Differential Equations*

This homework is about the basic techniques of solving differential equations. It spans four lectures: (i) History of ODEs, (ii) Solving first-order ODEs, (iii) Solving second-order constant coefficient homogeneous ODEs, (iii) Linear independence and Wronskians.

Instructions:
Homework should be completed on loose-leaf paper stapled or bound together.
Write your name and class neatly at the top or on a separate title page.
Presentation matters and 5% of your mark will be on how you present your solutions.

Hand-in date: Wednesday Feb. 22, 2012

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1. **Classification of Differential Equations**

   For each of the following differential equations, (i) find their order and (ii) determine whether they're linear or nonlinear.

   (a) $(1 + 2y^2)y' - y \cos x = 0$

   (b) $y''' + x^2 y = 0$

   (c) $(y')^2 + xy = 0$

   (d) $y'' + 4y' + 4y = 0$

   (e) $y'' - xy = 0$

   (f) $y''y' - x = 0$

   (g) $y'' - \frac{(x^2+2)}{2y'} = 0$

   (h) $y''' + xy^2 = 0$

   (i) $x^2 y'' + 2xy' + 2y - x^2 = 0$

   (j) $y'' + y' + y = 0$

2. **First Order ODEs**

Find the general solution to the following ODEs

(a) $x \log x y' + y = x$

(b) $y'' = 2(y')^2 x$

(c) $y' = y \log y$

(d) $y' = y^{1+\varepsilon}$ where $\varepsilon > 0$.

(e) $y' = e^x - y \log x$

3. **Substitutions for first-order problems**

Sometimes, a first order problem may not be separable, but by making a smart variable substitution, it can be changed to a separable ODE (of a new function). For example, a **homogeneous first order** ODE is one of the form

$$\frac{dy}{dx} = F\left(\frac{y}{x}\right),$$

which can be solved by setting $v(x) = y(x)/x$.

Use this method to compute the orthogonal trajectories of the family of curves $x^2 + y^2 = cx$ where $c \in \mathbb{R}$. Make a sketch of both sets of trajectories.

4. **Direction fields**

For both problems: (i) sketch the direction field using a grid of points in $-3 \leq x \leq 3$ and $-3 \leq y \leq 3$; (ii) within your direction field, trace out the approximate solution which corresponds to the specified initial condition; (iii) solve the initial value problem analytically.

(a) $y' = x - y + 1$ with $y(-3) = 0$

(b) $y' = x^2 - y - 2$ with $y(0) = -1$

5. **Formula for integrating factors**

Show that the general solution to the ODE

$$\frac{dy}{dx} + P(x)y = Q(x),$$

on some interval, $x \in I$, where P and Q are continuous, is given by

$$y(x) = e^{-\int^x P(s)\,ds}\left[\int^x Q(s)e^{\int^s P(t)\,dt}\,ds + C\right]$$

where C is constant.

(Notice the careful use of integration limits and dummy indices!)

6. **Second Order Homogeneous ODEs**

   For each of the following ODEs, find the solution corresponding to the initial conditions $y(0) = 0$ and $y'(0) = 1$.

   (a) $y'' + y' - 6y = 0$

   (b) $y'' + 6y' + 9y = 0$

   (c) $y'' + y' + 1.25y = 0$

   (d) $y'' + 2y' + y = 0$

   (e) $y'' + 4y' + 5y = 0$

7. **Reduction of order**

   For the following ODEs, you are given one of the solutions. Use the method of reduction of order in order to solve for the general solution.

   (a) $x^2y'' + 2xy' - 2y = 0$ for $x > 0$, and given $y_1 = x$.

   (b) $xy'' - y' + 4x^3y = 0$ for $x > 0$, and given $y_1 = \sin x^2$

8. **Higher-order linear constant coefficient differential equations**

The work required in solving and analyzing an n^{th} order, linear, constant coefficient ODE, compared to the 2^{nd} order variety, is somewhat like the work in solving an $N \times N$ matrix compared to solving a 2×2 matrix: *much of the theory remains the same, with the exception of the nasty algebra.*

(a) Given the n^{th} order ODE

$$\mathcal{L}[y] = a_n y^{(n)} + a_{n-1} y^{(n-1)} + \ldots + a_1 y' + a_0 y = 0,$$

where the coefficients, $a_k \in \mathbb{R}$, for $k = 1, \ldots, n$, use the ansatz $y = e^{rx}$ and write down the characteristic equation for r. Write down the general solution in the case where all the roots, λ_k, are distinct (you may leave complex exponentials as they are).

(b) If we write D for a differential operator d/dx, note that the differential equation can be written as

$$\mathcal{L}[y] = \left(a_n D^n + a_{n-1} D^{n-2} + \ldots + a_1 D + a_0\right) y = 0.$$

For simplicity, assume that the characteristic equation has one distinct root, $r = r_0$, and $p = n - 1$ repeated roots, $r = r_1$. Explain why the differential equation can be written as

$$\mathcal{L}[y] = (D - r_0)(D - r_1)^p y = (D - r_1)^p (D - r_0) y = 0.$$

Conclude that every solution of the equation

$$(D - r_1)^k y = 0,$$

must be a solution of the ODE.

(c) By using reduction of order, and setting $y = u(x)e^{r_1 x}$, show that the missing p solutions, which correspond to the repeated root, is given by

$$y(x) = ue^{r_1 x} = \left(c_1 + c_2 x + c_3 x^3 + \ldots + c_p x^{p-1}\right) e^{r_1 x}. \tag{1}$$

(d) Explain how the above work can be extended to show that, given any characteristic equation with a repeated root of multiplicity p [that is, the factor $(r - r_1)^p$ appears in the characteristic equation, and p is the largest such number], the general solution corresponding to $e^{r_1 x}$ is given by (1).

9. **A higher order boundary value problem (to infinity!)**

Consider the ODE

$$y''' = -y.$$

where the $x \in [0, \infty)$, and $y \in \mathbb{R}$.

(a) Write down the general solution, making sure the final result has real (unknown) coefficients and quantities.

(*Hint:* you will have to remember how to take complex-valued roots)

(b) This is a third order problem, so you would expect three boundary conditions. Suppose you are only given two conditions:

$$\begin{cases} y(0) = 1 \\ y(\infty) < \infty. \end{cases}$$

Can you directly verify that these two conditions are all that is required in order to produce a particular solution of the ODE?

10. **Linear independence and the Wronskian**

(a) Determine whether the set $\{x^3, |x^3|\}$ is linearly dependent on $[-1, 1]$

(b) Find $W(x^3, |x|^3)(x)$ on $[-1, 1]$

(c) In class, you learned about a theorem which concludes that if the Wronskian of two functions is zero, then the functions are linearly dependent. Why do the results from the previous two parts not contradict this theorem?

11. Matlab: Numerical derivatives:

Print your code and output.
Make sure that it is legible (comments are good!).
The easiest way to do this is to go to FILE » PUBLISH MATLAB FILE

In class, we showed how numerical derivatives can be computed using first order forwards or backwards differences. In practice, it is usually better to use **centered finite differences**. If $[x_{i-1}, x_i, x_{i+1}]$ are three neighboring, equally spaced grid points (with similar notations for the function values), then the centered difference is defined by:

$$(\delta f)_i = \frac{f_{i+1} - f_{i-1}}{2h}, \tag{2}$$

where h is the discretization size, $h = x_{i+1} - x_i = x_i - x_{i-1}$.

(a) Use the function $f(x) = \sin x$ over the domain $x \in [0, 6\pi]$. Discretize the function over this interval using $N = 20$ points, and plot the function. Label the x and y axes.

(b) Use the centered difference formula (still with $N = 20$) to compute the derivative of f. Use forwards and backwards differences at the first and last mesh point. Plot a graph of the derivative using a dashed line and compare with the exact derivative using a solid line.

(c) Use different values of N and show using a log log plot that the formula (2) makes an error of $\mathcal{O}(h^2)$ [read: of "order" h^2] for our particular function. You can define the error in your discretized functions as

$$\text{error} = ||\mathbf{f} - \mathbf{g}||_\infty = \max\{|f_1 - g_1|, |f_2 - g_2|, \ldots, |f_N - g_N|\},$$

that is to say, the error is defined as the maximum over all the deviations at each grid point.