350 Introduction to Differential Equations

1. Use Euler's Method to find the numerical solution to the logistic initial value problem

$$\frac{dy}{dx} = \frac{1}{3}y(8-y), \qquad y(0) = 1,$$

within the interval $x \in [0, 5]$, and using $n = 8, 16, 32$ points. Make a plot of the exact solution and each of the three numerical approximations (using different line styles). Find the three values for the error,

$$\text{error} = ||y_{\text{exact}} - y_{\text{approx}}||_{\infty}.$$

2. Use Euler's method to approximate the solution of the initial value problem

$$\frac{dy}{dx} = y\cos x, \qquad y(0) = 1$$

within the interval $x \in [0, 6\pi]$, and using $n = 50, 100, 200, 400$ points. Make a plot of the exact solution and each of the three numerical approximations (using different line styles). Find the values of each of the errors, and make a loglog plot of the error as a function of the step size. Based on this figure, what is the order of the error? Can you make a conjecture on how many points, approximately, it would take in order to achieve 12 digits of accuracy?

3. Return to the previous question and solve the exact same problem, but this time using the Improved Euler's method. Can you make a conjecture on how many points, approximately, it would take in order to achieve 12 digits of accuracy?

   Bonus: Use the tic, toc command to time your runs. Based on your previous conjecture, and the amount of time it took to complete $n = 400$ runs, how long would you expect for it to take in order to achieve 12 digits of accuracy?

4. Use (the regular) Euler's method to approximate the solution of the initial value problem

$$\frac{dy}{dx} = x^2 + y^2, \qquad y(0) = 1,$$

on the interval $x \in [0, 1]$. Begin with a step size of $h = 0.1$, and perform computations also for step sizes of $h = 0.02$ and $h = 0.005$. Make a plot of each of these numerical computations. Can you make a conjecture about the nature of the solution, and in particular, the region of existence?

5. The differential equation

$$\frac{dy}{dt} = ky(M-y) - h\sin\left(\frac{2\pi t}{P}\right),$$

models a population that is periodically harvested and restocked with period P and maximal harvesting/restocking rate h. By using different initial populations, $y(0)$, and investigating the numerical solution for $t > 0$ (using the improved Euler's method), can you describe how the eventual population behaves based on the initial population?