# EXISTENCE AND UNIQUENESS

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There are all sorts of existence and uniqueness results for ODEs, and some require more stringent conditions than others. In this lecture, we will prove and discuss the local version of Picard's Theorem, which is one of the more standard results.

The key idea to proving existence is to recast the search for a solution in terms of an iterative process. In fact, we did this very thing last class in using Euler's method to approximate numerical solutions. Our derivation of the Improved Euler's method hinged upon the recasting of the differential into an integral equation. However, whereas the numerical algorithms we developed relied upon a marching approximation to the solution (whereby the approximation at a given point is used to follow the approximation to the next point), the method we introduce in this lesson—Picard's Method—uses a more global method. Here, we initiate the algorithm with a function defined over an interval, and the program produces a more accurate approximation over this interval.

Beginning with the initial value problem

$$y' = f(x,y) \qquad y(x_0) = y_0,$$

then assuming that a solution to the nonlinear first order equation exists, then we can integrate

$$y(x) = y(x_0) + \int_{x_0}^{x} f(t, y(t))\, dt. \tag{9.1}$$

This is known as an **integral equation** (notice that the unknown, $y(x)$, appears on both sides). You can verify that by taking the derivative of this expression, then by the Fundamental Theorem of Calculus, we recover the original differential equation, so (9.1) is indeed a solution. Thus, there is a solution of (??) if and only if there is a solution of (9.1).

Now suppose that we began with an initial guess; we might as well take the constant solution, $y_0$ as the initial guess. Were we to substitute this into the integral equation, the right hand-side would not quite equal the left hand-side (unless $y_0$ is actually a solution). The idea behind Picard's Method is to use the left hand-side as a *better* approximation of the solution. Thus, we have a $y_1$, with

$$y_1 = y_0 + \int_{x_0}^{x} f(t, y_0)\, dt. \tag{9.2}$$

The procedure can be repeated. Thus, we may write

$$y_n = y_0 + \int_{x_0}^{x} f(t, y_{n-1}(t))\, dt, \tag{9.3}$$

for $n \geq 1$. Next we look at an example for which this iterative procedure is carried out.

*Example* 9.1. Consider the IVP

$$y' = x + y \quad \text{with } y(0) = 0.$$

Find the associated integral equation which the solution must satisfy and compute the first three successive approximations $y_n$, $n = 1, 2, 3$ starting with $y_0 = y(0) = 0$. Based on this, deduce the form of the general term $y_n$, find the limiting function $y(x) = \lim_{n\to\infty} y_n(x)$ and verify that this is indeed the solution to the IVP.

Integrating both sides gives

$$\int_0^x y'(t)\, dt = \int_0^x [t + y(t)]\, dt,$$

thus we have

$$y_n(x) = \int_0^x [t + y_{n-1}(t)]\, dt.$$

If $y_0 = 0$, we then have

$$\begin{aligned}
y_1 &= \int_0^x [t + 0]\, dt = \frac{x^2}{2} \\
y_2 &= \int_0^x \left[t + \frac{t^2}{2}\right] dt = \frac{x^2}{2} + \frac{x^3}{6} \\
y_3 &= \int_0^x \left[t + \frac{t^2}{2} + \frac{t^3}{6}\right] dt = \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} \\
\ldots &= \ldots \\
y_n &= \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \ldots + \frac{x^{n+1}}{(n+1)!}
\end{aligned}$$

Thus we have

$$\lim_{n\to\infty} y_n = \sum_{n=2}^{\infty} \frac{x^n}{n!} = \sum_{n=0}^{\infty} \frac{x^n}{n!} - 1 - x = e^x - 1 - x = y(x).$$

We can verify that this is indeed the correct solution by substitution (instead, we could have also tackled the ODE using integrating factors).

Now in practice, nobody(?) really solves differential equations using the method of Picard iterates, primarily because it tends to be slowly convergent and not computationally efficient. The advantage, however, is that it is a powerful theoretical tool, which can be used to prove the existence of a solution to nonlinear ODEs, under certain conditions.

### *9.0.1 Basic ideas behind the existence proof*

The big question is whether or not the iterates of Picard's method converges, and if so, whether it converges to the solution of the ODE. Consider the difference between two neighboring iterates:

$$y_{n+1}(x) - y_n(x) = \int_{x_0}^{x} [f(t, y_n(t)) - f(t, y_{n-1}(t))] \, dt. \tag{9.4}$$

Now if in fact the iterates converge to some limit, then a necessary condition is for the difference between adjacent iterates to tend to zero as $n \to \infty$. We take the modulus of this equation, and remember that the modulus of an integral is smaller than the integral of the modulus of the function. Then

$$\left| y_{n+1}(x) - y_n(x) \right| \leq \left| \int_{x_0}^{x} |f(t, y_n(t)) - f(t, y_{n-1}(t))| \, dt \right|.$$

The issue here is that we now need to bound the integrand. This motivates the following definition:

**Definition 9.1** (Lipschitz condition). *A function, $g(x)$, continuous on the set, $U$, satisfies a Lipschitz condition if there exists a fixed positive $A$ such that*

$$|g(x_2) - g(x_1)| \leq A|x_2 - x_1|, \tag{9.5}$$

*holds for all pairs of points, $x_1$ and $x_2 \in U$. The constant, $A$, is then called the Lipschitz constant.*

Being Lipschitz is a stronger condition than being continuous, but is a weaker condition than being continuously differentiable. To see this, notice that if the derivative, $g'(x)$, exists as a bounded function on $U$, then the Mean Value Theorem can be applied to show that

$$|g(x_2) - g(x_1)| = \left| g'(x^*)|x - x^*| \right| \leq P|x - x^*|,$$

for some bound, $P > 0$, and some point, $x^*$ between $x_1$ and $x_2$. Thus, all differentiable functions (with a bounded derivative) are Lipschitz.

When $f(x, y) = y(x)$ (*i.e.* a one-dimensional function), then the Lipschitz condition can be intuitively checked by creating a cone bounded by two lines of slope $A$ and $-A$ in the $(x, y)$ plane, and verifying that all relevant points within the domain, $U$, are bounded by the cone (see Figure [—]). Here are some examples of functions that are or are not Lipschitz.

*Example* 9.2. Consider the function

$$g(x) = x^2.$$

Since $g(x)$ is continuously differentiable everywhere, then it is Lipschitz on any closed or open set, *except* for any set which includes infinity, such as $[0, \infty)$. Why? Because the slope of the graph tends to infinity as $|x| \to \infty$. In order to prove this more formally, we can always set $x_1 = 0$ and $x_2 = n$, and note that

$$\frac{f(x_2) - f(x_1)}{x_2 - x_1} = \frac{n^2}{n} = n \to \infty,$$

as $n \to \infty$, so the quantity can never be bounded by a fixed number.

*Example* 9.3. Consider the function

$$g(x) = |x|,$$

on the interval $[-1, 1]$. Notice that $g$ is not differentiable at $x = 0$. It is, however, Lipschitz, since the absolute value of the slope between any two points is at most equal to one. Thus, the Lipschitz constant can be set to $A \geq 1$.

*Example* 9.4. Consider the function defined by

$$g(x) = \sqrt{x},$$

on the interval $[0, 1]$. The function is continuous everywhere, but not differentiable at $x = 0$. Is it Lipschitz? We would not expect so, since the slope tends to infinity at $x = 0$. Formally, we can select $x_1 = 0$ and $x_2 = n$, and examine the limiting behaviour as $n \to 0$. We then have

$$\frac{f(x_2) - f(x_1)}{x_2 - x_1} = \frac{\sqrt{n}}{n} = \frac{1}{\sqrt{n}} \to \infty,$$

as $n \to 0$. So there can not be a Lipschitz constant.

For our purposes, however, we need a slightly different Lipschitz condition, since we are dealing with a multivariate function.

**Definition 9.2** (Lipschitz condition of the second argument). *A function, $f(x, y)$, continuous on the set, $U$, satisfies a Lipschitz condition (of the second argument) if there exists a fixed positive A such that*

$$|f(x, y_2) - f(x, y_1)| \leq A|y_2 - y_1|, \tag{9.6}$$

*holds for all pairs of points, $(x, y_1)$ and $(x, y_2) \in U$. The constant, A, is then called the Lipschitz constant.*

However, the same intuitive picture holds in order to ensure a Lipschitz condition of the second argument: for every $x$ in $U$, we take a slice of the surface, $f(x, y)$, parallel to the $y$-axis. Then for each pair of points $(x, y_1)$ and $(x, y_2)$, we are required to bound the slope of the secant line by the *same* Lipschitz constant.

Lipschitz is the key condition which assures existence, for it allows us to force the iterates of Picard's method to converge. We are now ready for the theorem and its proof.

## 9.1 Picard's Theorem

We are given the following initial value problem:

$$y' = f(x, y) \qquad y(x_0) = y_0.$$

Let us confine $(x, y) \in R$, where $R$ is the rectangle defined by

$$R = \{(x, y) : |x - x_0| \leq h \text{ and } |y - y_0| \leq k\},$$

where we have chosen $h$ (or $k$) such that

$$h \leq \frac{k}{M} \quad \text{where} \quad M = \max_{(x,y) \in R} |f(x, y)|.$$

Next, let us denote a function $f(x, y(x))$ which is both continuous (in $x$) and Lipschitz (in $y$) as $f \in (C, Lip)$.

**Theorem 9.1.** (Picard's Theorem) *If $f \in (C, Lip)$ in $R$, then there exists a unique solution to the initial value problem, valid for $(x, y) \in R$.*

*Proof.*
**Step 1.** Re-write as an integral iteration:

$$y_0(x) = y_0 \quad \text{and} \quad y_{n+1}(x) = y_0 + \int_{x_0}^{x} f(t, y_n(t))\, dt.$$

**Step 2.** We need the iterates to converge. That is, if we set $e_0 = e_0$ and $e_n = y_n - y_{n-1}$, then notice

$$y_n = \sum_{k=0}^{n} e_k,$$

and we need the series to converge to a unique limit. In particular, the differences must tend to zero, $e_n \to 0$ as $n \to \infty$. Notice that

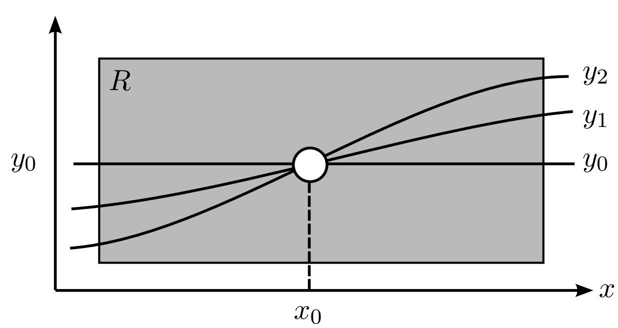
$$\begin{aligned} |e_{n+1}(x)| &= \left| \int_{x_0}^{x} \{f(t, y_n) - f(t, y_{n-1})\}\, dt \right| \\ &\leq \left| \int_{x_0}^{x} |f(t, y_n) - f(t, y_{n-1})|\, dt \right|. \end{aligned} \tag{9.7}$$

In order to bound the behaviour of $e_{n+1}$, we need to bound the behaviour of the integrand. That's why the Lipschitz condition becomes important.

**Step 3**. Let's show that each $y_n$ is continuous and its graph lies within $R$. Continuity is easy since $f$ is continuous. $(x, y_{n+1}) \in R$ follows because

$$|y_{n+1} - b| \leq \left| \int_{x_0}^{x} |f(t, y_n)|\, dt \right| \leq M|x - x_0| \leq Mh \leq k,$$

so $y_n$ always leaves the rectangle through the sides (see below).



**Step 4.** Now we show $e_n$ gets smaller. From (9.7) and assuming Lipschitz constant $K$,

$$|e_{n+1}(x)| \le K\left|\int_{x_0}^{x} |y_n - y_{n-1}|\,dt\right| \le K\left|\int_{x_0}^{x} |e_n(t)|\,dt\right| \tag{9.8}$$

How does the sequence look? With $e_0 = y_0$, then

$$|e_1(x)| = |y_1 - y_0| = \left|\int_{x_0}^{x} f(t, y_n)\,dt\right| \le M|x - x_0|.$$

Inserting this into (9.8) gives

$$|e_2(x)| \le \frac{MK}{2}|x - x_0|^2,$$

and so on, with

$$|e_n(x)| \le \frac{MK^{n-1}}{n!}|x - x_0|^n \le \frac{MK^{n-1}}{n!}h^n.$$

This is enough to guarantee (by the M-test) that the series converges uniformly and so the limit $y_n \to y(x)$ exists and is continuous as $n \to \infty$.

**Step 5**. Is this solution unique? Suppose we have two of them: $y(x)$ and $Y(x)$ and consider their difference:

$$\begin{aligned}
|e(x)| &= \left|\int_{x_0}^{x} \{f(t, y) - f(t, Y)\}\,dt\right| \\
&\le \left|\int_{x_0}^{x} |f(t, y) - f(t, Y)|\,\,dt\right| \\
&\le K\left|\int_{x_0}^{x} |y - Y|\,\,dt\right| \\
&= K\left|\int_{x_0}^{x} |e(t)|\,\,dt\right|.
\end{aligned}$$

But $e(x)$ is continuous in $R$, so it must be bounded, say $|e(x)| \le B$. Then

$$|e(x)| \le BK|x - x_0|.$$

But if we keep iterating this procedure, we get

$$|e(x)| \le \frac{BK^n}{n!}|x - x_0|^n \le \frac{BK^n}{n!}h^n \to 0,$$

as $n \to \infty$. So the difference between $y$ and $Y$ is zero.

□

## 9.2 Picard's Theorem for first-order systems

The same ideas which we used to prove Picard's Theorem for the case of first-order nonlinear ODEs, the same can be used to prove the version for first-order systems. For example, consider the case of the initial value problem:

$$\frac{d\mathbf{x}}{dt} = \mathbf{F}(t, \mathbf{x}), \qquad \mathbf{x}(t_0) = \mathbf{x}_0,$$

where

$$\mathbf{x}(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} \qquad \mathbf{F}(t, \mathbf{x}) = \begin{pmatrix} f(t, \mathbf{x}) \\ g(t, \mathbf{x}) \end{pmatrix}$$

We would then define the Picard iterates as

$$\mathbf{x}_{n+1} = \mathbf{x}_0 + \int_{t_0}^{t} \mathbf{F}(s, \mathbf{x}_n(s))\, ds. \tag{9.9}$$

This will converge as before if we have the Lipschitz condition,

$$||\mathbf{F}(t, \mathbf{u}) - \mathbf{F}(t, \mathbf{v})|| \leq A||\mathbf{u} - \mathbf{v}||,$$

for some appropriate definition of "size" for a vector, which could be, for example,

$$\left|\left|\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}\right|\right| = |u_1| + |u_2|.$$