7 QUALITATIVE METHODS

Early on, you had already worked with drawing direction fields for the study of first-order ODEs. This yielded a simple and powerful way of studying ODEs qualitatively. The goal of this lecture is to develop similar qualitative methods for the study of two coupled first-order nonlinear ODEs, of the form

$$\frac{d\mathbf{x}}{dt} = \begin{bmatrix} f(x_1, x_2) \\ g(x_1, x_2) \end{bmatrix},$$

where $\mathbf{x}(t) = [x_1(t)\ x_2(t)]^T$. Notice that the rate of change of the solutions, $\mathbf{x}(t)$ only depends on time implicitly through the functional behavior of $x_1(t)$ and $x_2(t)$. These type of equations are called **autonomous** ODEs. Such problems effectively allow us to reduce the dimensionality of the problem by one; so rather than having to plot trajectories in x_1x_2t-space, we only need to understand how solutions look in the x_1x_2-plane (which is called the **phase plane**. Because we are operating in $\mathbb{R}^2$ (for the dependent variables), qualitative methods can be easily visualized. The techniques developed here have important implications for the study of more complicated systems (*e.g.* higher-dimensional systems, non-autonomous systems), but these may not be visualized so easily.

Before we study phase planes, however, we will look at phase *lines*.

7.1 A GUIDED EXAMPLE OF A PHASE-LINE ANALYSIS

Consider the following equation

$$\dot{y} = r\left(1 - \frac{y}{K}\right)y, \tag{7.1}$$

with $y(t) > 0$ for $t > 0$, as well as $r, K > 0$. This equation is called **autonomous** because it does not explicitly involve the independent variables, x, in the rate-of-change equation (notice that if x were to be replaced by t for 'time', this would mean that the rate of change at any point in time does not depend on how it had evolved to that state).

Instead of solving this nonlinear equation (how?), we can analyse the dynamics qualitatively. The points where $y' = 0$ are called **fixed points** (or **equilibrium points** or **steady state** points)—these are at

$$y = 0 \text{ and } y = K.$$

If the solution begins at either of these two points, it must remain here for all time. We plot a graph of y vs. $f(y)$ (Fig. 7.1, left). Along the y-axis, we draw arrows to indicate the trajectories of the solution. For $0 < y < K$, $y' = f(y)$ is positive, so the solution must grow. For $y > K$, $y' = f(y) < 0$ so the solution must decay. Thus $y = 0$ is an *unstable* fixed point, and $y = K$ is an *asymptotically stable* fixed point. Note that

$$y'' = \frac{df(y)}{dt} = \frac{df}{dy}\frac{dy}{dt} = f'(y)f(y),$$

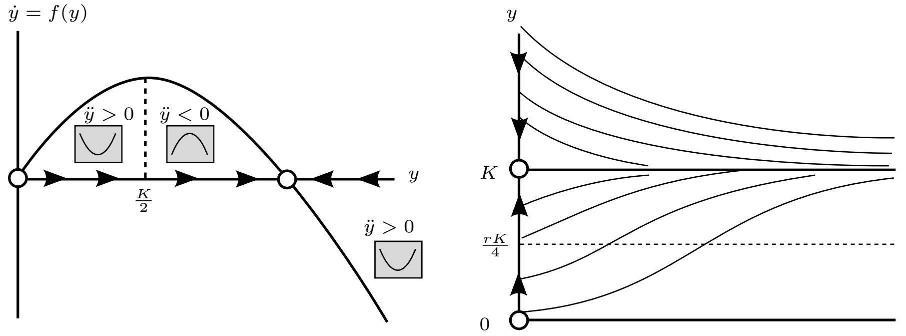


Figure 7.1: Phase line and solutions for the problem in eqn (7.1).

so we can also judge the concavity of $y(t)$ based on the slope and sign of $f(y)$. This is added to the figure. Finally, we can sketch the solutions $y(t)$, given different initial conditions (Fig. 7.1, right). We follow with another example of this one-dimensional phase analysis.

Example 7.1. Consider the following equation

$$\dot{y} = -r\left(1 - \frac{y}{T}\right)\left(1 - \frac{y}{K}\right)y,$$

where $y(t) > 0$ for $t > 0$, with $r > 0$ and $0 < T < K$.

Recall that the process is:

- Determine the fixed points.
- Plot $f(y)$ and use arrows along the y-axis to indicate the trajectories of solutions along the phase line. Determine the sign of $\ddot{y}$ along the phase line.
- Sketch the solutions in the (t, y) plane.

There are fixed points at $y = 0, T$, and K. The sketch of $f(y)$ and the (t, y) plane are shown in Figure 7.2.

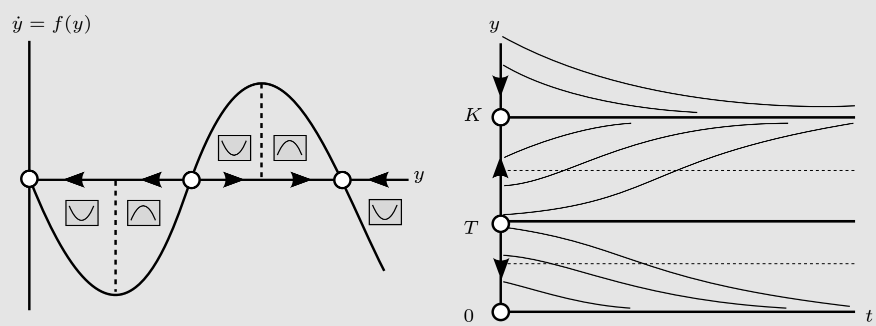


Figure 7.2: Phase line (left) and solutions (right) for the example.

Let us now move from phase lines to phase spaces, and in particular, let us examine the constant-coefficient linear system

$$\frac{d\mathbf{x}}{dt} = A\mathbf{x} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \mathbf{x}, \tag{7.2}$$

which we now know well from the previous lecture. Let λ_1 and λ_2 be the two eigenvalues of A, and we suppose that there exists two linearly independent eigenvectors, given by $\mathbf{v}_1$ and $\mathbf{v}_2$. The general solution can then be written as

$$\mathbf{x}(t) = c_1\mathbf{v}_1 e^{\lambda_1 t} + c_2\mathbf{v}_2 e^{\lambda_2 t},$$

This expression indicates that each solution trajectory, $x(t)$, can be expressed in terms of the directions indicated by the eigenvectors (we call these the **eigendirections**). The case when there is only one linearly independent eigenvector will be addressed in your problem set.

The key simplification in this problem relates to the autonomous nature of the ODE; because the local rate of the solution at the point (x_1, x_2), does not depend on time, then we can completely describe the solution trajectories in the x_1x_2-plane (the **phase plane**). Moreover, in the problem $\dot{\mathbf{x}} = A\mathbf{x}$, there is a fixed point at $\mathbf{x} = 0$, and our interest is to characterize the solutions near this fixed point.

7.2.1 *Real unequal eigenvalues of the same sign (improper nodes)*

Let us assume that the eigenvalues, λ_1 and λ_2 are both real and negative, with $\lambda_2 < \lambda_1 < 0$. If $\mathbf{v}_1$ and $\mathbf{v}_2$ are the associated eigenvectors, the general solution is given by

$$\mathbf{x} = c_1\mathbf{v}_1 e^{\lambda_1 t} + c_2\mathbf{v}_2 e^{\lambda_2 t}.$$

Consider the limit of $t \to \infty$. The key is to note that if both eigenvalues are negative, the solution decays to zero as t increases, but with $e^{\lambda_1 t}$ becoming exponentially larger than $e^{\lambda_2 t}$ (since $\lambda_1 > \lambda_2$). Thus, as $t \to \infty$, the solution becomes tangential to $\mathbf{v}_1$ (the *slow eigendirection*):

$$\mathbf{x}(t) \sim c_1\mathbf{v}_1 e^{\lambda_1 t},$$

and merges into the origin. If we reverse the direction of time, then as $t \to -\infty$, the solution becomes dominated by the faster eigendirection, i.e. goes tangential to $\mathbf{v}_2$ (Fig. 7.3, left). Thus, in the case when both eigenvalues are strictly negative, then the critical point is called a **stable node**. Similarly, if both eigenvalues are positive, then all the directions are reversed, and so the critical point is called an **unstable node**.

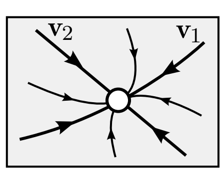


Figure 7.3: Stable node

7.2.2 Eigenvalues real and of opposite sign (saddle point)

Assume λ_1 and λ_2 are real and of opposite signs, with $\lambda_1 < 0 < \lambda_2$. Repeating the previous argument, we see that as $t \to \infty$, the solutions approaches $\mathbf{x} \sim c_2\mathbf{v}_2 e^{\lambda_2 t}$, and tends towards the line spanned by the vector $\mathbf{v}_2$. This gives a **stable saddle** On the other hand, as $t \to -\infty$, the solutions approach the line spanned by $\mathbf{v}_1$. This gives an **unstable saddle** (Fig. 7.4).

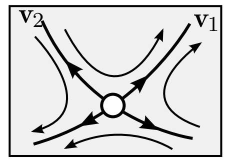


Figure 7.4: Saddle point

7.2.3 Centers and spirals (complex eigenvalues)

Assume that λ_1 and λ_2 are complex eigenvalues, and write $\lambda_{1,2} = \alpha \pm i\omega$ with $\omega \neq 0$. Solutions are given by

$$\mathbf{x} = c_1\mathbf{v}_1 e^{\alpha t} e^{i\omega t} + c_2\mathbf{v}_1 e^{\alpha t} e^{-i\omega t}.$$

As we showed earlier, this solution will contain multiples of oscillations given by

$$e^{\alpha t}[\cos(\omega t) \pm i \sin(\omega t)] \, .$$

If $\alpha = 0$, these oscillations will be of constant amplitude, so in the phase plane, the trajectories will be closed orbits around the origin (**centers**). If $\alpha < 0$, the amplitude of the oscillations will decay to zero, so instead of closed orbits, the trajectories will be spirals tending to zero (**stable spiral**). And if $\alpha > 0$, the amplitude of the oscillations will grow to infinity, and the phase picture will be **unstable spirals**.

7.2.4 Stability and instability

Based on the behaviour of the different types for the constant coefficient system of ODEs, we propose the following definition in order to to classify that different stability types.

Definition 7.1. *Recall that the critical points of the nonlinear system of ODEs, $\dot{\mathbf{x}} = \mathbf{F}(\mathbf{x})$ are given by the points, $\mathbf{x} = \mathbf{x}^*$, where $\mathbf{F}(\mathbf{x}^*) = 0$. The point $\mathbf{x}^*$ is **stable** if, given $\epsilon > 0$, there exists $\delta > 0$ such that every solution $\mathbf{x} = \phi(t)$ of the ODE which at $t = 0$ has*

$$||\phi(0) - \mathbf{x}^*|| < \delta$$

exists for all positive t and with

$$||\phi(t) - \mathbf{x}^*|| < \epsilon,$$

and thus guarantees that if ϕ starts sufficiently close to $\mathbf{x}^*$, then it remains close to $\mathbf{x}^*$ for all time. Moreover, the point $\mathbf{x}^*$ is ***asymptotically stable*** *if there exists δ_0 with $0 < \delta_0 < \delta$ such that*

$$||\phi(0) - \mathbf{x}^*|| < \delta_0 \Rightarrow \lim_{t\to\infty} \phi(t) = \mathbf{x}^*,$$

so that, solutions that start sufficiently close to the fixed point, tend towards the fixed point as time progresses.

7.3 NONLINEAR AUTONOMOUS ODES

The constant coefficient matrix equations studied in the last section are fine, but a more general problem we'd like to tackle is the solution of the nonlinear autonomous problem:

$$\dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}), \tag{7.3}$$

where for simplicity, we shall restrict $\mathbf{x} \in \mathbb{R}^2$. In principle, we are concerned with when we can approximate the behaviour of (7.3) near the equilibrium points using a matrix equation with constant coefficients:

$$\dot{\mathbf{x}}(t) = A\mathbf{x}, \tag{7.4}$$

where $A \in \mathbb{R}\times\mathbb{R}$. From the previous section, we shall introduce the following terminology:

Definition 7.2. *A point, $\mathbf{x}_0 \in \mathbb{R}^2$ is called an* ***equilibrium point*** *(or saddle point or fixed point) of (7.3) if $\mathbf{f}(\mathbf{x}_0) = 0$. The equilibrium point is called a* ***hyperbolic*** *equilibrium point if none of the eigenvalues of $D\mathbf{F}(\mathbf{x}_0)$ have a zero real part. The linear system (7.4), with $A = D\mathbf{F}(\mathbf{x}_0)$ is called the* ***linearization*** *of (7.3).*

In the same way that the equation of a line (the first two terms of the Taylor series) can be used to approximate any function near a point, *except* when the function behaves like a quadratic, a cubic, etc. – we can linearize the nonlinear ODE, and apply the techniques from the previous section to determine the nature of the critical point. The key question to answer, however, is how do we know when a nonlinear critical point is linearizable and when it is not?

Let us assume that there is a fixed point at $\mathbf{x}_0 = \mathbf{0}$. We can always shift a non-zero equilibrium point to the origin by introducing the new independent variable, $\hat{\mathbf{x}} = \mathbf{x} - \mathbf{x}_0$, so this is done without loss of generality. We now Taylor series expand:

$$\dot{x} = f(\mathbf{0}) + \frac{\partial f(\mathbf{0})}{\partial x}x + \frac{\partial f(\mathbf{0})}{\partial x}y + \mathcal{O}\left(x^2, y^2\right).$$

The first term on the right disappears by definition of a fixed point. The symbol $\mathcal{O}$ refers to terms that involve quadratic or higher-order powers. Because x and y are already small, such powers will (hopefully) be negligible. The same thing is done with $\dot{y} = g$. The nonlinear system can be approximated by

$$\begin{bmatrix}\dot{x}\\ \dot{y}\end{bmatrix} \sim D\mathbf{F}(0,9)\begin{bmatrix}x - x_0\\ y - y_0\end{bmatrix},$$

where $D\mathbf{F}$ is the **Jacobian matrix**:

$$A(x,y) = \begin{bmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{bmatrix}.$$

We shall not prove this here, but it can be shown that the approximation (7.4) is a good first approximation to the full nonlinear problem near the equilibrium point, so long as $D\mathbf{F}(0)$ does not contain any eigenvalues with a zero real part. Essentially, this is because if there exists an eigenvalue with a zero real part, then any small perturbation (*i.e.* the higher-order terms we neglected) is enough to shift the eigenvalue off the imaginary axis and change the behaviour near the equilibrium point in a drastic manner. We will demonstrate this intuitively.

Example 7.2. Consider the differential equation

$$\begin{aligned} \dot{x} &= f(x(t), y(t)) = -x + x^3 \\ \dot{y} &= g(x(t), y(t)) = -2y, \end{aligned}$$

and use a local analysis near the equilibrium points to study the problem qualitatively.

Before we get started, we introduce the useful idea of the x-nullclines (where $\dot{x} = 0$) and y-nullclines (where $\dot{y} = 0$). The x-nullclines are given by $x = 0$ and $x = \pm 1$. Along here, the trajectory vectors point straight upwards. The y-nullcline is given by $y = 0$; here, the trajectory vectors horizontally.

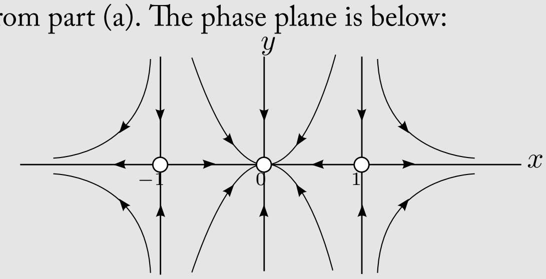
The Jacobian is given by

$$A = \begin{bmatrix} -1 + 3x^2 & 0 \\ 0 & -2 \end{bmatrix}$$

so we have for the two critical points:

$$A(0,0) = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix} \qquad A(\pm 1, 0) = \begin{bmatrix} 2 & 0 \\ 0 & -2 \end{bmatrix}.$$

The linearised system near $(0,0)$ has two negative eigenvalues and so is a stable node, while near $(\pm 1, 0)$, there is one positive and one negative eigenvalue, so the fixed point is a saddle. We can easily verify that the eigenvectors in both cases are oriented along the x and y-axis; this should be obvious from part (a). The phase plane is below:



Example 7.3. Consider the differential equation

$$\dot{x} = -y + ax(x^2 + y^2) \tag{7.5}$$

$$\dot{y} = x + ay(x^2 + y^2) \tag{7.6}$$

with $a \in \mathbb{R}$.

The only fixed point is $(0, 0)$. By linearising the system near this point, we get the linearized problem

$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} \sim \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

so the eigenvalues are $\lambda = \pm i$ and it appears that $(0, 0)$ is a *stable center*. Is this really the behaviour of the nonlinear problem?

Let us make the transformation to polar coordinate, and write $x = r\cos\theta$ and $y = r\sin\theta$. Then we can write (7.5) as

$$\frac{d}{dt}(r\cos\theta) = -r\sin\theta + ar^3\cos\theta.$$

By the product rule

$$\dot{x} = \frac{d}{dt}(r\cos\theta) = \dot{r}\cos\theta - \dot{\theta}r\sin\theta,$$

thus,

$$\dot{r}\cos\theta - \dot{\theta}r\sin\theta = -r\sin\theta + ar^3\sin\theta. \tag{7.7}$$

Similarly, from (7.6), we get

$$\dot{r}\sin\theta + \dot{\theta}r\cos\theta = r\cos\theta + ar^3\sin\theta. \tag{7.8}$$

To solve the above two equations for $\dot{r}$ and $\dot{\theta}$, we multiply the first by $\cos\theta$, multiply the second by $\sin\theta$, and add and subtract the resultant equations. The result is

$$\dot{r} = ar^3$$

$$\dot{\theta} = 1$$

The equation in θ simply tells us that trajectories encircle the origin clockwise at unit angular velocity. If $\alpha = 0$, then the radial velocity remains zero at all times, so the trajectories are simply closed circles around the origin (a center). If $\alpha > 0$, the radial velocity increase (exponentially), so the result is an unstable spiral. If $\alpha < 0$, the result is a stable spiral.

So we conclude that the linearised problem does not accurately predict the nonlinear problem. The key is that the small nonlinear terms we neglected are enough to turn a center into a spiral; after all, to do so only requires us to alter the trajectories by an infinitesimal amount (enough so that the closed loops no longer close). In general, linearisation may fail for borderline cases: when the eigenvalues are both purely imaginary, or when there are two equal eigenvalues.