INHOMOGENEOUS AND HIGHER ORDER ODES

5

Recall from the last lecture that with

$$\mathscr{L}[y] = y'' + P(x)y' + Q(x)y, \tag{5.1}$$

where the coefficients, $P(x)$ and $Q(x)$ are continuous on some interval, $x \in \mathscr{I}$, we have

$$\mathscr{H} \quad \text{(homogeneous)} : \mathscr{L}[y] = 0, \tag{5.2}$$
$$\mathscr{N} \quad \text{(inhomogeneous)} : \mathscr{L}[y] = R(x), \tag{5.3}$$

Definition 5.1 (FSS). *Recall that the general solution of $\mathscr{H}$ is given by $y = c_1y_1 + c_2y_2$, where $\{y_1, y_2\}$ have a non-zero Wronskian (and are thus linearly independent). Then we call y_1 and y_2 a **fundamental set of solutions** (FSS) of $\mathscr{H}$.*

Property 5.1. *If Y_1 and Y_2 are solutions of $\mathscr{N}$, then $Y_1 - Y_2$ is a solution of $\mathscr{H}$. Thus, if $\{y_1, y_2\}$ is a fundamental set of solutions, then*

$$Y_1 - Y_2 = c_1y_1 + c_2y_2.$$

Proof. Note that

$$\mathscr{L}[Y_1 - Y_2] = \mathscr{L}[Y_1] - \mathscr{L}[Y_2] = f(x) - f(x) = 0.$$

□

Theorem 5.1 (General solution of the inhomogeneous problem). *The general solution of $\mathscr{N}$ can be written as*

$$y(x) = \Big[c_1y_1(x) + c_2y_2(x)\Big] + y_p(x),$$

*where y_1 and y_2 are solutions of $\mathscr{H}$, c_1 and c_2 are arbitrary constants, and $y_p(x)$ is any **particular solution** of $\mathscr{N}$.*

Proof. Follows directly from the previous property. □

Thus, in order to solve the inhomogeneous problem, we need to only find a particular solution, and then add it to the general solution of the homogeneous problem. Finding the particular solution of the inhomogeneous problem may not be trivial, but we shall introduce two methods: (i) the method of undetermined coefficients, and (ii) the method of variation of parameters.

5.1 METHOD OF UNDETERMINED COEFFICIENTS

This method applies to linear, inhomogeneous, constant coefficient ODEs. We'll do three examples with different forcing functions.

Example 5.1. Consider the particular solution of

$$y'' - 3y' - 4y = 3e^{2t}.$$

Our goal is to find a function, $y = y_p$, such that, when substituted into $y'' - 3y' - 4y$, returns the forcing function, $3e^{2t}$. Recall that when we substitute a function like e^{2t}, derivatives reproduce the exponential. Thus, we try the ansatz $y_p = Ae^{2t}$. Substituting into the ODE gives

$$4Ae^{2t} - 3 \cdot 2Ae^{2t} - 4Ae^{2t} = 3e^{2t},$$

or solving for A gives

$$A = -\frac{1}{2}.$$

We can verify that the general solution of the homogeneous equation is given by $y_h = c_1e^{4t} + c_2e^{-t}$. Thus, by Theorem 5.1, the general solution of the full inhomogeneous equation is

$$y = c_1e^{4t} + c_2e^{-t} - \frac{1}{2}e^{2t}.$$

Example 5.2. Find the particular solution of the ODE:

$$y'' - 3y' - 4y = 2\sin t.$$

In this case, we recall that since, derivatives of $\sin t$ produce $\sin t$ and $\cos t$, then a suitable guess might be a linear combination of these two functions. Trying $y = A\sin t + B\cos t$ gives,

$$\Big[-A\sin t - B\cos t\Big] - 3\Big[A\cos t - B\sin t\Big] - 4\Big[A\sin t + B\cos t\Big] = 2\sin t.$$

Then, matching terms with sin and cos give the system of two equations:

$$-5A + 3B = 2 \tag{5.4}$$

$$-3A - 5B = 0, \tag{5.5}$$

which yields $A = -5/17$ and $B = 3/17$. Thus, the particular solution is given by $y_p = -5/17\sin t + 3/17\cos t$. If we combine this with the homogeneous solution from the previous example, we get for the general solution,

$$y = c_1e^{-4t} + c_2e^{t} + \left[-\frac{5}{17}\sin t + \frac{3}{17}\cos t\right].$$

Notice that if we have the inhomogeneous, constant coefficient problem:

$$\mathscr{L}[y] = f_1(x) + f_2(x) + \ldots + f_n(x),$$

where $n \geq 1$, and if we know that the solution of the individual problems,

$$\mathscr{L}[y] = f_i(x),$$

is given by $y = y_p^{(i)}$, for $1 \le i \le n$, then by the linearity of the differential problem, the solution of the complete problem is

$$y_p = y_p^{(1)} + y_p^{(2)} + \ldots + y_p^{(n)}.$$

Example 5.3. Consider the ODE

$$y'' - 3y' - 4y = 3e^{2t} + 2\sin t.$$

From the previous remark, and the two first examples, the general solution is given by

$$y = c_1 e^{-4t} + c_2 e^t - \frac{1}{2}e^{2t} + \left[-\frac{5}{17}\sin t + \frac{3}{17}\cos t\right].$$

Example 5.4. Consider the ODE

$$y'' + 9y = t^2 e^{3t}.$$

The forcing function of the right hand-side, and the presence of the $9y$ term on the left hand-side indicates that we need a term like t^2e^{3t} in the solution. However, derivatives of this term also produce terms like e^{3t} and te^{3t}. We thus try something like

$$y = (At^2 + Bt + C)e^{3t}.$$

Substituting this ansatz into the ODE gives

$$\left[9At^2+(12A+9B)t+(2A+6B+9C)\right]e^{3t}+9(At^2+Bt+C)e^{3t} = t^2e^{3t},$$

and so we must solve

$$\begin{array}{llll} 18A & & & = 1 \\ 12A & +18B & & = 0 \\ 2A & +6B & +18C & = 0, \end{array}$$

which gives $A = 1/18$, $B = -1/27$, and $C = 1/162$. Thus, one particular solution is given by

$$y_p = \left[\frac{1}{18}t^2 - \frac{1}{27}t + \frac{1}{162}\right]e^{3t}.$$

5.2 VARIATION OF PARAMETERS

Variation of parameters is a powerful method that gives **any** particular solution to $\mathscr{L}[y] = f(x)$. This will method will be demonstrated in class. (MISSING NOTES)

Higher-order problems, such as the linear n^{th} order ODE:

$$P_n(t)y^{(n)} + P_{n-1}(t)y^{(n-1)} + \ldots + P_1(t)y' + P_0(t)y = f(t), \qquad (5.6)$$

share many similar methods and approaches to the second-order problems we have studied thus far. For example, for the case of the constant coefficient, homogeneous ODE:

$$p_n y^{(n)} + p_{n-1} y^{(n-1)} + \ldots p_1 y' + p_0 y = 0, \qquad (5.7)$$

we would again use the ansatz, $y = e^{rt}$, which yields the characteristic equation

$$Z(r) \equiv p_n r^n + p_{n-1} r^{n-1} + \ldots p_1 r + p_0 = 0.$$

The solution of the characteristic equation then gives the the n roots (perhaps repeated and complex-valued), which yield the general solution of the homogeneous problem. Thus we see that the theory which we have developed for second-order problems with constant coefficients should hold for the most part, except that the algebra may get much more involved for higher-order problems.

Example 5.5. Consider

$$y'' + Py' + Qy = 0,$$

where P and Q are constant, and $y = y(t)$. Let us make the substitutions:

$$\begin{aligned} x_1 &= y \qquad \Rightarrow \qquad \dot{x_1} = x_2 \\ x_2 &= y' \qquad \Rightarrow \qquad \dot{x_2} = -Px_2 - Qx_1. \end{aligned}$$

Thus, we can write the ODE in terms of the system of equations

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -q & -p \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix},$$

or with $\mathbf{x} = [x_1\, x_2]^T$, and the matrix A defined in the obvious way, we simply have the matrix equation:

$$\dot{\mathbf{x}} = A\mathbf{x}.$$

From the last example, we see that a useful (and often, essential) trick is to reduce the n^{th} order problem (5.6) to a first-order system of differential equations. To do that, we introduce a new function for each derivative of $y = y(t)$:

$$\begin{aligned} x_1 &= y \qquad \Rightarrow \qquad \dot{x_1} = x_2 \\ x_2 &= \dot{y} \qquad \Rightarrow \qquad \dot{x_2} = x_3 \\ &\vdots \qquad\qquad\qquad\qquad \vdots \\ x_n &= y^{(n-1)} \qquad \Rightarrow \qquad \dot{x_n} = \ldots, \end{aligned}$$

The last entry follows from using the original ODE. Assuming that $p_n(t) \neq 0$ within the domain of interest, then we define $a_i(t) = -p_i(t)/p_n(t)$, giving

$$\dot{x_n} = a_{n-1}(t)x_n + \ldots + a_1(t)x_2(t) + a_0(t)x_1(t).$$

Combining the above into a system, we get the matrix equation

$$\dot{\mathbf{x}} = \mathbf{A}(t)\mathbf{x}(t) + \mathbf{F}(t), \tag{5.8}$$

where

$$\mathbf{x}(t) = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix}$$

$$\mathbf{A}(t) = \begin{bmatrix} 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 \\ a_0(t) & a_1(t) & a_2(t) & a_3(t) & \cdots & a_{n-1}(t) \end{bmatrix}$$

$$\mathbf{F}(t) = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ f(t) \end{bmatrix}$$

Thus we see that the n^{th} order problem for $y = y(t)$ has now been turned into a first order system of differential equations for the scalar function, $\mathbf{x} = [x_1\, x_2\, \ldots\, x_n]$, for which $\mathbf{x} : \mathbb{R} \mapsto \mathbb{R}^n$. In the next lecture, we will look towards understanding how such matrix equations are solved.