SECOND-ORDER DIFFERENTIAL EQUATIONS II

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In the last lecture, we showed on second-order linear differential equations with constants coefficients can be solved. Now that you understand the intuitive ideas behind such ODEs, it's time to go back and clean up the theory. We'll begin with introducing a shorthand. The second-order linear operator, $\mathscr{L}$, is a map of functions, with

$$\mathscr{L}[y] = y'' + P(x)y' + Q(x)y, \tag{4.1}$$

where the coefficients, $P(x)$ and $Q(x)$ are continuous on some interval, $x \in \mathscr{I}$. Because the linearity of differentiation, then this operator is **linear**, with

$$\mathscr{L}[\alpha y_1 + \beta y_2] = \alpha\mathscr{L}[y_1] + \beta\mathscr{L}[y_2],$$

for α and $\beta \in \mathbb{R}$, and functions y_1 and y_2. Thus, $\mathscr{L}$ is a linear transformation on a suitable vector space of differentiable functions. We now definite the homogeneous problem, $\mathscr{H}$, and the inhomogeneous problem, $\mathscr{N}$, with

$$\mathscr{H} \quad \text{(homogeneous)} : \mathscr{L}[y] = 0, \tag{4.2}$$

$$\mathscr{N} \quad \text{(inhomogeneous)} : \mathscr{L}[y] = R(x), \tag{4.3}$$

where R is a given function. Note that

Property 4.1 (Vector space of homogeneous solutions). *The homogeneous solutions of $\mathscr{H}$ form a vector space.*

Proof. Most of the required properties of a vector space are easily shown. Notice that the zero, $y \equiv 0$ is a solution of $\mathscr{H}$, and also that if y_1 and y_2 are solutions, then $\alpha y_1 + \beta y_2$ is a solution, for $\alpha, \beta \in \mathbb{R}$. □

In the following, we will often use $\mathscr{H}$ to correspond to *both* the homogeneous problem and also the vector space of solutions which correspond to the homogeneous problem.

4.1 LINEAR INDEPENDENCE AND THE WRONSKIAN

Property 4.1 assures us that, given any two solutions, y_1 and y_2, we can construct more solutions by forming linear combinations. Of course, the natural question to ask is whether these linear combinations span the solution set, that is, whether any arbitrary solution, $y(x)$, can be represented as a linear combination of y_1 and y_2. For this, we will need to return to the concept of linear independence.

Definition 4.1 (Linear independence). *The set, $S = \{f, g\}$, is linearly dependent on $x \in \mathscr{I} = [a, b]$ if and only if there are real constants, c_1 and c_2, not all zero, such that*

$$c_1 y_1 + c_2 y_2 = 0, \tag{4.4}$$

for all $x \in \mathscr{I}$. Note that in the following, we will sometimes omit the statement that $x \in \mathscr{I}$, when writing (4.4).

Let us now review some additional facts about solving matrix equations. Consider the 2×2 system

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = A\mathbf{x} = 0.$$

Remember from your linear algebra that this equation has a non-trivial solution if and only if $|A| \neq 0$.

Definition 4.2 (Wronskian). *If f and g are differentiable functions on $\mathscr{I} = [a, b]$, the Wronskian, $W(f, g)(x)$, of f and g is defined by*

$$W(f,g)(x) = \begin{vmatrix} f & g \\ f' & g' \end{vmatrix} = fg' - gf'.$$

The Wronskian will be particularly important to the idea of solutions of $\mathscr{H}$, and in fact, it will turn out that given two solutions, either W is identically zero in $\mathscr{I}$ or W is never zero in $\mathscr{I}$.

Theorem 4.1 (Linear independence of differentiable functions). *If f and g are differentiable and linearly dependent functions on $\mathscr{I} = [a, b]$, then the Wronskian, $W(f, g)$ is identically zero on $\mathscr{I}$.*

Proof. By linear dependence, we have

$$c_1 f + c_2 g = 0,$$

for some c_1 and c_2, not both zero. With this in mind, assuming that the functions are differentiable, we form the two equations

$$\begin{matrix} c_1 f + c_2 g \\ c_1 f' + c_2 g' \end{matrix} = \begin{pmatrix} f & g \\ f' & g' \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = 0.$$

By assumption, c_1 and c_2 are not both zero, thus, by our knowledge of linear algebra, the determinant of the matrix is zero. Thus $W(f, g)(x) = 0$ for all $x \in \mathscr{I}$. □

Example 4.1. Does the converse of the theorem hold? That is, if $W \equiv 0$ for all $x \in I$, are f and g linearly dependent?

4.2 LINEAR INDEPENDENCE FOR SECOND-ORDER LINEAR ODES

There is a slight problem with what we will now present, and this relates to the following theorem, which we hinted in the previous lecture:

Theorem 4.2 (Existence and Uniqueness). *Consider the initial value problem*

$$y'' + P(x)y' + Q(x)y = R(x), \quad \textit{for } x \geq x_0 \tag{4.5}$$
$$y(x_0) = y_0 \quad \textit{and} \quad y'(x_0) = y_0', \tag{4.6}$$

where P, Q, and R are continuous on the open interval, $x \in I$. Then there is exactly one solution to the IVP, and moreover, this solution exists in the interval I.

Essentially, we will need to make use of this theorem (which is proven in next week's lectures). I've reversed the order of presentation because I think it's best to give a more concrete overview of differential equations, before having to deal with existence and uniqueness results. The alternative is to present the heavy theory before you have had some experience with solving ODEs; while this is more mathematically elegant, it's also perhaps more difficult.

Theorem 4.3 (Abel's Theorem). *Let y_1 and y_2 be solutions of the homogeneous equation*

$$\mathscr{L}[y] = y'' + py' + qy = 0,$$

where p and q are continuous on I. The Wronskian is then given by

$$W = \textit{const} \times \exp\left[-\int^x p(t)\, dt\right].$$

Thus, W is either zero or always non-zero for $x \in I$.

Proof. Differentiate the Wronskian,

$$W' = y_1 y_2'' - y_2 y_1''.$$

Since y_1 and y_2 satisfy

$$y'' + p(x)y' + q(x)y = 0,$$

then substituting the values of the second derivatives into W' gives

$$W' = -p(x)\Big[y_1 y_2' - y_2 y_1'\Big] = -pW.$$

This is a separable ODE, which we can simply solve to give the result. Because the exponential term is strictly positive, then either $W \equiv 0$ for all $x \in I$ (*i.e.* $c = 0$), or W is always non-zero. □

We have the last all-important theorem:

Theorem 4.4. *If $S = \{y_1, y_2\}$ is a solution set of $\mathscr{H}$. Then y_1 and y_2 are linearly dependent on $\mathscr{I}$ if and only if $W(y_1, y_2)(x) = 0$ for all $x \in \mathscr{I}$. Alternatively, y_1 and y_2 are linearly independent if and only if $W(y_1, y_2)(x)$ is never zero in the interval.*

Proof. If y_1 and y_2 are linearly dependent then by Theorem 4.1, the Wronskian is zero for all x. It remains to prove the reverse direction. We first let x_0 be a point such that the Wronskian is non-zero. Then we have the fact that the determinant of the matrix in the equation,

$$\begin{pmatrix} y_1(x_0) & y_2(x_0) \\ y_1'(x_0) & y_2'(x_0) \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = 0,$$

is equal to zero. Thus, there exists non-trivial solutions, $c_1 = C_1$, and $c_2 = C_2$, not both zero. Consider now the function

$$\phi(x) = C_1 y_1 + C_2 y_2.$$

Clearly, $\phi(x_0) = 0$ and $\phi'(x_0) = 0$ under our choice of C_1 and C_2. Now return to the ODE problem, $\mathscr{H}$. One possible solution which satisfies the same conditions at x_0 is simply the trivial solution, with $\phi \equiv 0$ on $x \in \mathscr{I}$. However, by Theorem 4.2, this solution must be unique. Thus,

$$\phi = C_1 y_1 + C_2 y_2 = 0,$$

for $x \in \mathscr{I}$. Since C_1 and C_2 are not both zero, this proves that $\{y_1, y_2\}$ is a linearly dependent set. □

If we combine what we have discovered thus far, we come to the following conclusion: if y_1 and y_2 are two solutions of $\mathscr{H}$, then the following statements are equivalent:

- $\{y_1, y_2\}$ form a basis for the set of solutions of $\mathscr{H}$.
- $\{y_1, y_2\}$ form a linearly independent set
- $W(y_1, y_2)(x_0) \neq 0$ for some $x_0 \in I$
- $W(y_1, y_2)(x) \neq 0$ for all $x \in I$

That is, if we begin with any one of the four statements, then the other three follow immediately.

4.3 INHOMOGENEOUS EQUATIONS

We now address how to solve the inhomogeneous equation, $\mathscr{L}[y] = f(x)$, in (4.3). Our solution method hinges on the following theorem.

Property 4.2 (General solution of the inhomogeneous problem). *The general solution of $\mathscr{N}$ can be written as*

$$y(x) = \Big[c_1 y_1(x) + c_2 y_2(x)\Big] + y_p(x),$$

*where y_1 and y_2 are solutions of $\mathscr{H}$, c_1 and c_2 are arbitrary constants, and $y_p(x)$ is any **particular solution** of $\mathscr{N}$.*

Proof. Let Y and y_p be any two solutions of $\mathscr{N}$. Notice that

$$\mathscr{L}[Y - y_p] = f(x) - f(x) = 0,$$

so $Y - y_p$ must be a solution of the homogeneous problem, $\mathscr{H}$. However, any general solution of $\mathscr{H}$ can be written,

$$Y - y_p = c_1 y_1 + c_2 y_2.$$

□

In other words, this theorem states that in order to solve the inhomogeneous equation, we first solve for the general solution of the homogeneous equation, and then we solve for a particular solution of the inhomogeneous equation. The general solution of the inhomogeneous equation is then given by the sum of the two expressions. The trick, then, is to find a particular solution of the inhomogeneous equation; this, we will discuss in the next lecture.