LECTURE 23

CLASSIFICATION OF PDEs & NUMERICS

At last, we have been introduced to three cardinal linear, second-order PDEs in 2D. The homogeneous versions are:

$$
\begin{cases}
\text{(i) Wave equation: } & u_{tt} - c^2 u_{xx} = 0 \quad \text{(hyperbolic)} \\
\text{(ii) Heat equation: } & u_t - \gamma u_{xx} = 0 \quad \text{(parabolic)} \\
\text{(iii) Laplace's equation: } & u_{xx} + u_{yy} = 0 \quad \text{(elliptic)}
\end{cases}
$$

- These are the prototypical representatives of the three fundamental types of PDEs.
- Each type has distinct analytical features, physical models, and numerical methods.
- Hyperbolic & parabolic generally have $y = t$ = time, so have initial and initial-boundary conditions.
- Elliptic is generally associated with space, so depends on boundary conditions.
- Most PDEs (even higher-dimen.) fall into one of three types (or possibly combinations)

The full classification of linear, 2nd order, PDEs is as follows:

$$(*) \qquad \mathcal{L}[u] = Au_{xx} + Bu_{xy} + Cu_{yy} + Du_x + Eu_y + Fu = G$$

$A, \dots, G$ can be functions of (x,y) and $u = u(x,y)$.

Where not all $A, B, C = 0$.

The key quantity is the <u>discriminant</u> $\Delta = B^2 - 4AC$, which is defined by analogy to the discriminant for the conic sections of:

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0.$$

$$\Rightarrow \begin{cases} \text{(i) hyperbola when } \Delta > 0 \\ \text{(ii) parabola when } \Delta = 0. \\ \text{(iii) ellipse when } \Delta < 0. \end{cases}$$

PDEs of the type $(*)$ are defined in the same way, but note $\Delta = \Delta(x,y)$ and (ii) only is parabolic if $A^2 + B^2 + C^2 \neq 0$.

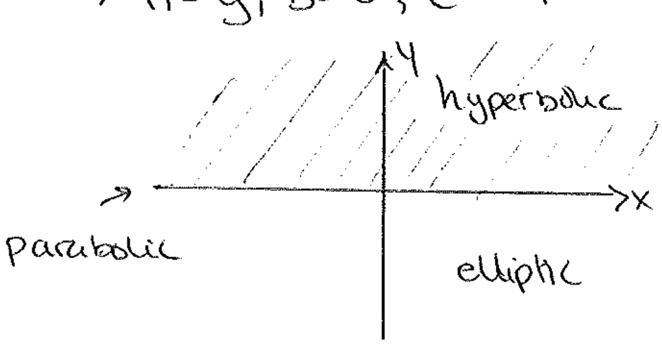
Example:

- Wave: $u_{xx} - u_{yy} = 0 \Rightarrow \Delta = 0^2 - 4(1)(-1) = 4 > 0$ $\Rightarrow$ hyperbolic
- Heat: $u_{xx} - u_y = 0 \Rightarrow \Delta = 0^2 - 4(1)(0) = 0 \Rightarrow$ parabolic
- Laplace: $u_{xx} + u_{yy} = 0 \Rightarrow \Delta = 0^2 - 4(1)(1) = -4 < 0 \Rightarrow$ elliptic

Notice that in some regions of (x,y)-space, the type can switch from one to another. The classic example is the Tricomi equation,

$$y\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial y^2} = 0.$$

$$\Rightarrow A = y,\ B = 0,\ C = -1 \Rightarrow \Delta(x,y) = 0^2 - 4(y)(-1)$$
$$= 4y.$$



This is a model for the shock wave that forms when an air-plane goes from subsonic (hyperbolic) to supersonic (elliptic)

For general quasi-linear (or completely nonlinear) PDEs, the classification may depend on (x,y), but also the particular solution u being considered.

CHARACTERISTICS

The notion of characteristics also allows us to classify PDEs. Consider $\mathcal{L}[u] =$ as before. We call $y = y(x)$ the characteristic curve of the PDE if

$$A(x,y)\left(\frac{dy}{dx}\right)^2 + B(x,y)\cdot\left(\frac{dy}{dx}\right) + C(x,y) = 0.$$

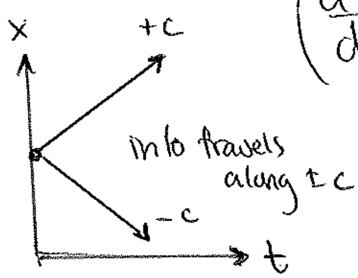
while if $x = x(y)$ is the characteristic:

$$A(x,y) - B(x,y)\cdot\left(\frac{dx}{dy}\right) + C(x,y)\left(\frac{dx}{dy}\right)^2 = 0.$$

In the wave eqn: $u_{tt} - c^2 u_{xx} = 0$

$\Rightarrow$ (with "t"="x", "x"="y")

$$\left(\frac{dx}{dt}\right)^2 - c^2 = 0 \Rightarrow \frac{dx}{dt} = \pm c$$



through each point ∃ two characteristics

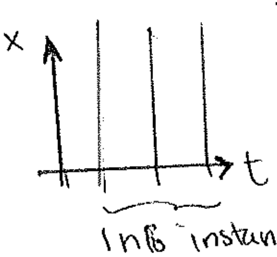
In Laplace equation: $u_{xx} + u_{yy} = 0$

$$\Rightarrow \left(\frac{dy}{dx}\right)^2 + 1 = 0 \Rightarrow \text{no characteristics.}$$

In heat eqn: $u_{xx} - u_t = 0.$

$$\Rightarrow \left(\frac{dt}{dx}\right)^2 = 0 \Rightarrow \text{characteristics are } t = \text{const.}$$

(vertical lines)



These properties hold because $(**)$ is quadratic eqn for $\left(\frac{dy}{dx}\right)$ and depends on $\Delta = B^2 - 4AC$.

Quick summary of differences between the three different types:

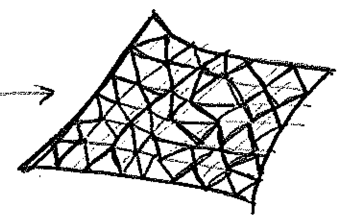
Property	Hyperbolic	Parabolic	Elliptic
(i) speed of propagation	Finite-speed, c	Infinite	
(ii) singularities $t>0$	Transported along characteristics	Lost immediately	◦ No characteristics
(iii) well-posed? $t>0$	yes	yes	◦ Solutions are always smooth regardless of B.C.s
(i) well-posed? $t<0$	yes	no.	◦ Localised disturbances felt everywhere
(v) $t \to \infty$	Solution does not decay	Solution decays	
(vi) Information	Transported along characteristics	Lost gradually	
	needs initial conditions	needs initial conditions	needs B.C.s.

Intro to numerics of PDEs.

Simplest approach: finite differences in time + space.

(Other approaches include finite elements $\Rightarrow$ solve mini-PDEs in triangular elements and couple everything

finite elements



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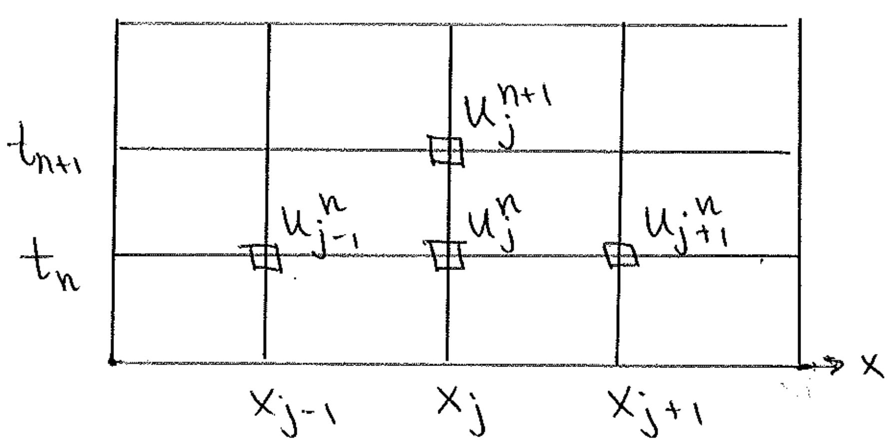
Consider heat equation:

$$\begin{cases} u_t = u_{xx} & -1 \le x \le 1 \\ u(-1,t) = 0 = u(1,t) \\ u(x,0) = u_0(x) \end{cases}$$

Set up a rectangular grid with $\Delta t = k$ and $\Delta x = h$ and with points: $x_0 = -1,\ x_1 = -1+h,\ \dots,\ x_N = 1-h,\ x_{N+1} = 1$



so $u(x_j, t_n) \approx u_j^n$

Recall that to approximate $u_{xx}(x_j, t_n)$, we can used centred difference

$$u_{xx}(x_j, t_n) \approx \frac{u_{j+1}^n - 2u_j^n + u_{j-1}}{h^2}$$

"Stencil"

$$\begin{matrix} & X & \\ & | & \\ X & — X — & X \end{matrix}$$

We now approximate the time derivative forwards in time (i.e. Euler's step)

$$u_t = u_{xx} \implies \frac{u_j^{n+1} - u_j^{n}}{k} = \frac{u_{j+1}^{n} - 2u_j^{n} + u_{j-1}^{n}}{h^2}$$

$$\implies u_j^{n+1} = \underset{\substack{\uparrow \\ b}}{\left(\frac{k}{h^2}\right)} u_{j+1}^{n} + \underbrace{\left(1 - \frac{2k}{h^2}\right)}_{a} u_j^{n} + \frac{k}{h^2} v_{j-1}^{n}$$

We can conside u^n as an N-vector and place in the form:

$$u^{n+1} = \underbrace{\begin{pmatrix} a & b & 0 & \cdots & 0 \\ b & a & b & & \vdots \\ & \ddots & \ddots & \ddots & \\ & b & a & b & \\ 0 & \cdots & & b & a \end{pmatrix}}_{\text{Tri-diagonal structure}} \begin{pmatrix} u_1^n \\ u_2^n \\ \vdots \\ u_{N-1}^n \\ u_N^n \end{pmatrix}$$

Note that the first coordinate is,

$$u_1^{n+1} = \frac{k}{h^2} u_2^n + \left(1 - \frac{2k}{h^2}\right) u_1^n + \frac{k}{h^2} \underset{\substack{\uparrow \\ =0}}{u_0^n}$$

$$= b u_2^n + a u_1^n$$

So the matrix takes care of the B.C.s $u_0^n = u_{N+1}^n = 0$.

(See Matlab code for this)

Numerical instability:

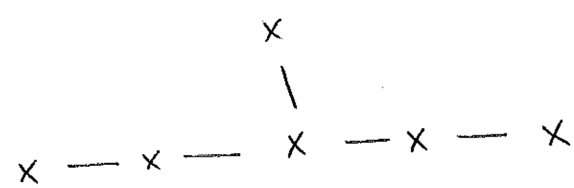
Consider 4^{th} order diffusion eqn

$$\begin{cases} u_t = -u_{xxxx} \\ u(-1) = u(1) = u'(-1) = u'(1) = 0 \\ \text{(minus sign in eqn for positive diffusion)} \end{cases}$$

Working out the finite differences:

$$\frac{u_j^{n+1} - u_j^n}{K} = -\frac{u_{j+2}^n - 4u_{j+1}^n + 6u_j^n - 4u_{j-1}^n + u_{j-2}^n}{h^4}$$

For a scheme like this:



Again, we need to solve $\vec{u}^{n+1} = A\vec{u}^n$

We see from Matlab code that numerical instability occurs unless K is tiny $(\lesssim 4.8 \times 10^{-8})$. Why?

Suppose @ time $t = t_n$, we have a sine wave:

$$u_j^n = e^{i\xi x_j} = e^{i\xi(hj)}$$

The numerical scheme demands:

$$u_j^{n+1} = u_j^n - \frac{K}{h^4}\left(u_{j+2}^n - 4u_{j+1}^n + 6u_j^n - 4u_{j-1}^n + u_{j-2}^n\right)$$

When we plug in the sine wave into this scheme we get:

$$u_j^{n+1} = g(\xi)\cdot e^{i\xi x_j}$$

and in order to avoid blow-up, we need $|g(\xi)| < 1$

After algebra, you can show:

$$g(\xi) = 1 - \frac{16k}{h^4}\sin^4\left(\frac{\xi h}{2}\right)$$

where ξ is a fixed wavenumber. Over all the possible wave numbers, we must have

$$1 - \frac{16k}{h^4} \leq g(\xi) \leq 1$$

and thus we need for stability

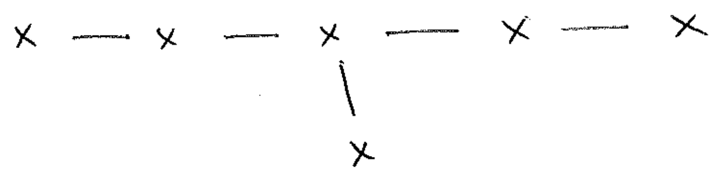
$$1 - \frac{16k}{h^4} \geq -1 \implies k \leq \frac{h^4}{8}$$

very tight! For $h = 0.025$, need $k \leq 4.883\times 10^{-8}$!

It's surprisingly easy to fix this instability which is present because the PDE is very stiff (stiff = widely different time scales are present)

To do this, we need IMPLICIT FORMULA:

$$u_j^{n+1} = u_j^n - \frac{K}{h^4}\left(u_{j+2}^{n+1} - 4u_{j+1}^{n+1} + 6u_j^{n+1} - 4u_{j-1}^{n+1} + u_{j-2}^{n+1}\right)$$



This is enough to guarantee the amplitude factor $g = g(\xi)$ has $|g(\xi)| \leq 1 \ \forall \xi$. At the linear algebra level, we need to solve:

$$B\vec{u}^{n+1} = \vec{u}^n \Rightarrow \vec{u}^{n+1} = B^{-1}\vec{u}^n$$

See Matlab, which is now stable for most K.