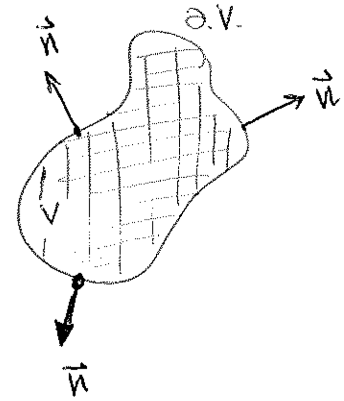
LECTURE 22
GREENS FUNCTIONS FOR PDES

We are now concerned with solving Poisson's equation with Dirichlet boundary conditions:

$$\nabla^2 u = -f(\vec{x}), \quad \vec{x} \in$$
$$u(\vec{x}) = h(\vec{x}), \quad \vec{x} \in \partial$$



The key to extending our ideas in 1D to 3D is GREEN'S SECOND IDENTITY:

$$\boxed{\int_{\partial V} \left(u \frac{\partial v}{\partial n} - v \frac{\partial u}{\partial n} \right) \cdot dV = \int_{V} \left(u \nabla^2 v - v \nabla^2 u \right) \cdot dS}$$

where $n = \vec{n}$ denotes the outer normal along the boundary.

Pf: Note that the identity in 1D reduces to

$$\int_a^b (uv'' - vu'') \cdot dx = (uv' - vu')\Big|_a^b$$

which follows from integration by parts.

Note that $\nabla \circ (v \nabla u) = \nabla v \circ \nabla u + v \nabla^2 u$

$$\Rightarrow \int_V \nabla \circ (v \nabla u) \cdot dV = \int_V \nabla v \circ \nabla u \cdot dV + \int_V v \nabla^2 u \cdot dV$$

Using the divergence theorem on the left:

$$\int_V \nabla \cdot (v \nabla u)\, dV = \int_{\partial V} (v \nabla u) \cdot \vec{n} \cdot dS$$

$$= \int_{\partial V} v \cdot \frac{\partial u}{\partial n} \cdot dS.$$

$$\Rightarrow \int_{\partial V} v \frac{\partial u}{\partial n} \cdot dS = \int_V (\nabla v \cdot \nabla u) \cdot dV + \int_V v \nabla^2 u \cdot dV$$

Notice the symmetry of the middle term. Switching u and v and adding/subtracting

$$\Rightarrow \int_V (u \nabla^2 v - v \nabla^2 u) \cdot dV = \int_{\partial V} \left(u \frac{\partial v}{\partial n} - v \frac{\partial u}{\partial n} \right) dS$$

▨

Green's second identity allows us to repose the diff. eqn. in terms of an integral eqn. Recall from the 1D problem that the key idea is to consider the response of the PDE to a concentrated unit delta impulse @ a point $\vec{\xi} \in V$:

$$-\nabla^2 u = \delta(\vec{x} - \vec{\xi})$$

but what is this?

In the case of $\mathbb{R}^2$, we can define the $\delta(\vec{x}-\vec{\xi}) = \delta_{\vec{\xi}}$ function by:

$$\delta_{\vec{\xi}}(\vec{x}) = \delta(\vec{x}-\vec{\xi}) = \delta(x-\xi, y-\eta)$$

where $\vec{\xi} = (\xi, \eta)$ and where

$$\delta_{\vec{\xi}}(\vec{x}) = 0, \ \forall \vec{x} \neq \vec{\xi} \quad \text{and} \quad \iint_V \delta_{(\xi,\eta)}(x,y) \cdot dx\, dy = 1 \quad \text{if } \vec{\xi} \in V.$$

Like in 1D, we can define such a δ through limits of sequence of functions, for example:

$$\delta_0(x,y) = \lim_{n \to \infty} \frac{n}{\pi} e^{-n(x^2+y^2)}$$

Alternatively because double integrals are evaluated as repeated single integrals, we could view

$$\delta_{(\xi,\eta)}(x,y) = \delta_\xi(x)\,\delta_\eta(y) = \delta(x-\xi)\,\delta(y-\eta)$$

So we wish to solve for the Green's Function, $G_{\vec{\xi}}(\vec{x}) = G(\vec{x};\vec{\xi})$

$$\begin{cases} \nabla^2 G(\vec{x},\vec{\xi}) = -\delta_{\vec{\xi}}(\vec{x}) & \text{in } x \in V \\ G(\vec{x},\vec{\xi}) = 0 & \text{on } x \in \partial V \end{cases}$$

Now we apply Green's identity to u (the solution of Poisson's eqn) and $v = G$

$$\int_V \Big(u \underbrace{\nabla^2 G}_{=-\delta(\vec{x}-\vec{\xi})} - G \underbrace{\nabla^2 u}_{=-f(\vec{x})}\Big) \cdot dV = \int_{\partial V} \Big(u \frac{\partial G}{\partial n} - G \cdot \frac{\partial u}{\partial n}\Big) \cdot dS$$

$G = 0$ on ∂V

$u = h(\vec{x})$ on ∂V

$$\Rightarrow \int_V \big(-u\,\delta(\vec{x}-\vec{\xi}) + G f\big) \cdot dV = \int_{\partial V} u \cdot \frac{\partial G}{\partial n} \cdot dS$$

By the sifting property, $\displaystyle\int_V u\,\delta(x-\vec{\xi})\,dV = u(\vec{\xi})$

$$\therefore\ u(\vec{\xi}) = \int_V G(\vec{x},\vec{\xi})\, f(\vec{x}) \cdot - \int_{\partial V} u(\vec{x}) \cdot \frac{\partial G(\vec{x},\vec{\xi})}{\partial n} \cdot dS \qquad (*)$$

note u is known on the boundary.

Note that it can be proven that Greens functions are symmetric in their arguments:

$$G(x,y;\xi,\eta) = G(\xi,\eta;x,y)$$

by applying Green's identity to $G(\vec{x};\vec{\xi})$ and $G(\vec{\xi};\vec{x})$.

Reversing $\vec{x}$ with $\vec{\xi}$ in the $(*)$, we come to the

$$\boxed{u(\vec{x}) = \int_V G(\vec{x},\vec{\xi}) f(\vec{\xi}) \cdot d\vec{\xi} - \int_{\partial V} u(\vec{\xi}) \cdot \frac{\partial G(x,\vec{\xi})}{\partial n} \cdot dS.}$$

FREE SPACE GF.

The free space GF., $G_0(\vec{x}, \vec{\xi})$ is the Green's Function that corresponds to the problem where $\Omega = \mathbb{R}^2$ (or $\mathbb{R}^3$) and so there are no boundaries:

$$\nabla^2 G_0(\vec{x}, \vec{\xi}) = -\delta(\vec{x} - \vec{\xi})$$

We solve for G_0 in 2D.

The trick is that because of the lack of boundaries, we only expect $G_0(\vec{x}, \vec{\xi}) = G_0(r)$ where $r = \|\vec{x} - \vec{\xi}\| = \sqrt{(x-\xi)^2 + (y-\eta)^2}$.

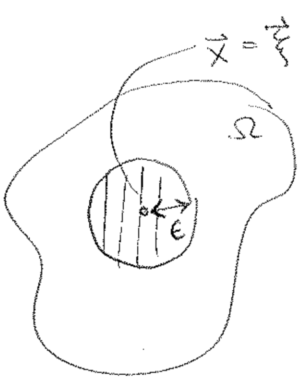
Away from $r = 0$, we must have

$$\nabla^2 G_0 = \frac{1}{r}\left(\frac{\partial}{\partial r}\left(r \frac{\partial G_0}{\partial r}\right)\right) = 0.$$

$$\Rightarrow G_0 = A + B \cdot \log r$$

We may discard A since only the singular part is important near the singularity. What is B?

$$\nabla^2 G_0 = -\delta(r) \Rightarrow \iint_{C_\epsilon} \nabla \cdot (\nabla G_0) \cdot r \cdot dr \cdot d\theta = -\int_{C_\epsilon} \delta(r) \cdot dx \cdot dy$$



$$\frac{\partial G_0}{\partial r} = \frac{B}{r} = \frac{B}{\epsilon}$$

$$\Rightarrow \int_0^{2\pi} \epsilon \cdot \frac{\partial G_0}{\partial r} \cdot d\theta = -1$$

$$\Rightarrow 2\pi B = -1 \Rightarrow B = \frac{-1}{2\pi}$$

The free space GFs for $\mathbb{R}^n$, are:

$$\boxed{G_0(\vec{x},\vec{\xi}) = \begin{cases} -\frac{1}{2}\|\vec{x}-\vec{\xi}\| & n=1 \\ -\frac{1}{2\pi}\log\|x-\vec{\xi}\| & n=2 \\ \frac{-1}{(2-n)A_n(1)}\cdot\|x-\vec{\xi}\|^{2-n} & n\geq 3 \end{cases}}$$

where $A_n(1)$ = area of unit sphere (e.g.

METHOD OF IMAGES.

The key is that G_0 characterizes the local singularity of the proper G.F. (that satisfies the correct B.C.s).

Thus, we write:

$$G = G_0(\vec{x},\vec{\xi}) + H(\vec{x},\vec{\xi})$$

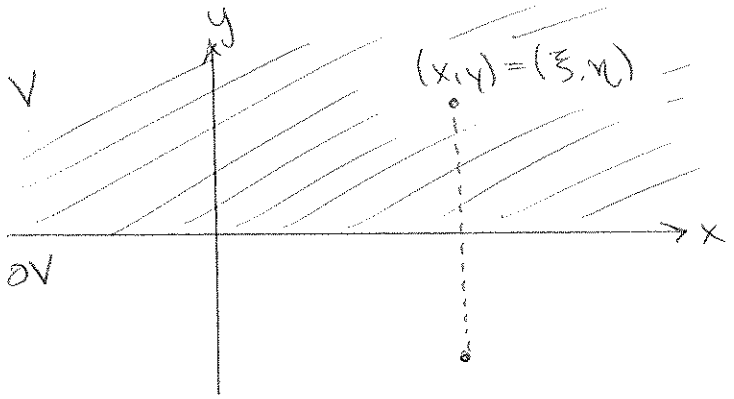
where H is any arbitrary solution to $\nabla^2 H = 0$.

Clearly, H will not change the fact $\nabla^2 G = -\delta(\vec{x}-\vec{\xi})$.

Moreover, we want to choose H so that it takes the value of $-G_0$ on ∂V. Then $G=0$ on ∂V.

This can be done using the method of images.

Suppose wish to solve for the G.F. in the UHP



We know:

$$G = -\frac{1}{2\pi}\log\left[(x-\xi)^2+(y-\eta)^2\right]^{2} + H(x,y;\xi,\eta)$$

$G_0 = -\frac{1}{2\pi}\log\|\vec{x}-\vec{\xi}\|$ corresponds to the response generated by placing a source @ $\vec{x}=\vec{\xi}$. In order for G to be zero on the boundary $y=0$, we place an opposite source @ the reflected point $\vec{\xi}^{*}=(\xi,-\eta)$.

$$G = -\frac{1}{4\pi}\log\left[(x-\xi)^2+(y-\eta)^2\right] + \frac{1}{4\pi}\log\left[(x-\xi)^2+(y+\eta)^2\right].$$

notice that $\nabla^2 G = -\delta(\vec{x}-\vec{\xi})$ since adding the reflected source does not change what occurs near $\vec{x}=\vec{\xi}$. (moreover, the reflected source solution is perfectly well behaved and diff. inside V). Also

$$G(x,0;\xi,\eta) = 0.$$

And this verifies G is the correct G.F.

Suppose we want to solve:
$$\begin{cases} \nabla^2 u = 0, & y>0. \\ u(x,0) = h(x) & \text{on } y=0. \end{cases}$$

We then have $$u(x,y) = -\int_{\partial V} h(\xi) \cdot \frac{\partial G(x,y;\xi,\eta)}{\partial n} \cdot dS$$

$$= -\int_{-\infty}^{+\infty} h(\xi)\,\underbrace{(-1)}_{\text{Since outer normal points down.}}\,\frac{\partial G(x,y;\xi,\eta)}{\partial y} \cdot dS$$

Since $\dfrac{\partial G}{\partial y} = \dfrac{1}{2\pi} \dfrac{(y-\eta)}{(x-\xi)^2 + (y-\eta)^2}$,

then $$u(x,y) = +\int_{-\infty}^{+\infty} h(\xi) \left(\frac{1}{2\pi}\right)\left(\frac{y}{(x-\xi)^2 + y^2}\right) \cdot d\xi.$$

GREENS FUNCTION FOR A SPHERE.

We would like to derive the GF. for the geometry of a unit circle, $V = \{\vec{x}, \|\vec{x}\| < 1\}$. We must then find the harmonic fn. $H(x,y)$ s.t.

$$G(x,y;\xi,\eta) = \frac{-1}{2\pi} \log\|\vec{x} - \vec{\xi}\| + H(\vec{x})$$

where $\vec{\xi} = (\xi,\eta) \in V$. We thus place the image source @ some unknown point $\vec{\xi}^* \in \mathbb{R}^2 \setminus \overline{V}$. The image source should have the form: $$H(\vec{x}, \vec{\xi}^*) = \frac{a}{2\pi} \log\|\vec{x} - \vec{\xi}^*\| + \frac{b}{2\pi}$$

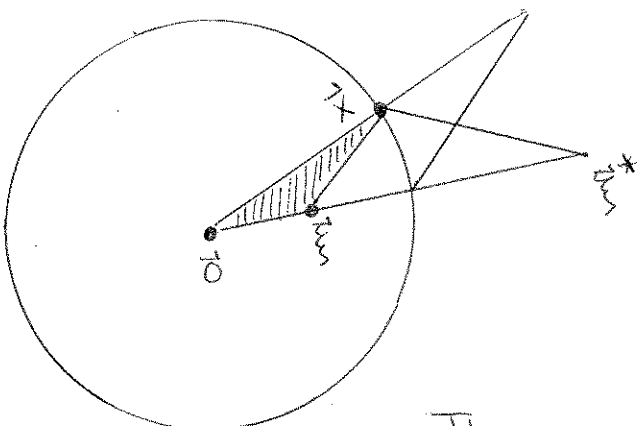
where a, b T.B.D.

We need the values on the boundary.

$$\log\|\vec{x}-\vec{\xi}\| = a\log\|\vec{x}-\vec{\xi}^*\| + b.$$

$$\Rightarrow \quad \|\vec{x}-\vec{\xi}\| = \lambda\|\vec{x}-\vec{\xi}^*\|^a \quad \text{for } \vec{x}\in\partial V. \qquad (*)$$

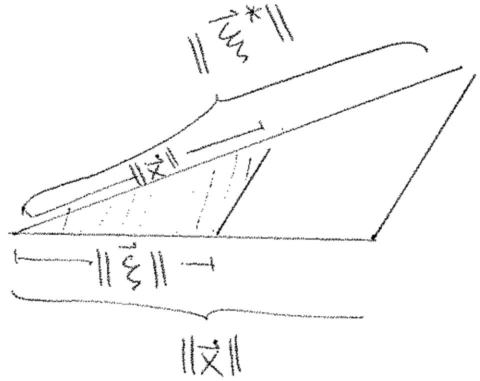
where $\lambda = e^b$.



The trick is to create similar triangles:

We assume $\vec{\xi}^*$ lies along the ray throug $\vec{\xi} \Rightarrow \vec{\xi}^* = c\,\vec{\xi}$.

Then we require $\Delta O\vec{x}\vec{\xi} \sim \Delta O\vec{\xi}^*\vec{x}$



Similarity requires:

$$\frac{\|\vec{\xi}\|}{\|\vec{x}\|} = \frac{\|\vec{x}\|}{\|\vec{\xi}^*\|} = \frac{\|\vec{x}-\vec{\xi}\|}{\|\vec{x}-\vec{\xi}^*\|} = \lambda$$

common ratio

Thus, we find that $a=1$, and since $\|\vec{x}\|=1$ on unit circle, $\|\vec{\xi}\| = \dfrac{1}{\|\vec{\xi}^*\|}$ and $\lambda = \|\vec{\xi}\|$.

This guarantees that $(*)$ holds and thus

$$G(\vec{x},\vec{\xi}) = -\frac{1}{2\pi}\log\|\vec{x}-\vec{\xi}\| + \frac{1}{2\pi}\log\|\vec{x}-\vec{\xi}^*\| + \frac{1}{2\pi}\log\|\vec{\xi}\|$$

and $\vec{\xi}^* = \dfrac{\vec{\xi}}{\|\vec{\xi}\|^2}$.