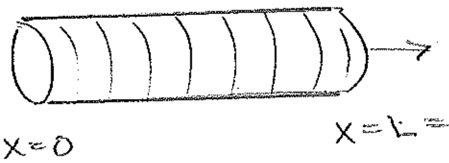
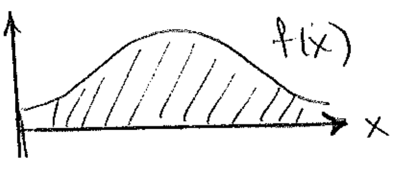
LECTURE 21: GREEN'S FUNCTIONS.

Consider the steady state heat distribution of a rod which is heated with some heat source density $f(x)$





$$\Rightarrow \begin{cases} u''(x) = -f(x) & 0 \le x \le 1 \\ u(0) = 0 \\ u(1) = 0 \end{cases}$$

Solving the B.V.P. :

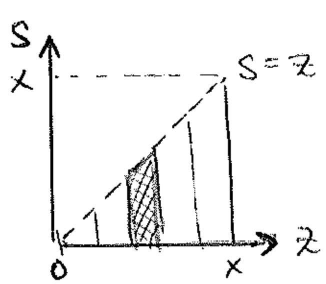
$$u(x) = -\int_0^x \int_0^z f(s) \cdot ds\, dz + x \int_0^1 \int_0^z f(s) \cdot ds \cdot dz$$

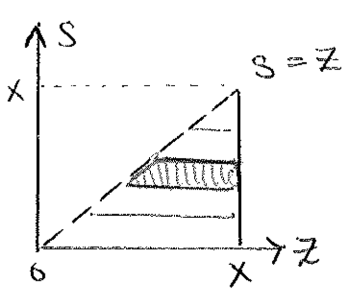
$$(*) \quad = -\int_0^x (x-s) f(s) \cdot ds + x \int_0^1 (1-s)\, f(s) \cdot ds$$

$$= -\int_0^1 \begin{Bmatrix} 0 & x \le s \\ (x-s) & x > s \end{Bmatrix} f(s) \cdot ds + x \int_0^1 (1-s) f(s) \cdot ds$$

$$= \int_0^1 \begin{Bmatrix} x(1-s) & x \le s \\ s(1-x) & x > s \end{Bmatrix} f(s) \cdot ds.$$

(*) Note $\displaystyle\int_0^x \int_0^z f(s) \cdot ds \cdot dz = \int_0^x \int_s^x f(s) \cdot dz \cdot ds$





- Let us imagine that $f(x)$ is a single "point" heat source in the middle of the rod @ $x = 1/2$. We could postulate that $f(x)$ is given by a delta function:

$$\begin{cases} \delta(x) = 0 \text{ if } x \neq 0. & (1a) \\ \int_{-\infty}^{+\infty} \delta(x) \cdot dx = 1 & (1b) \end{cases}$$

→ total "heat" is unity.

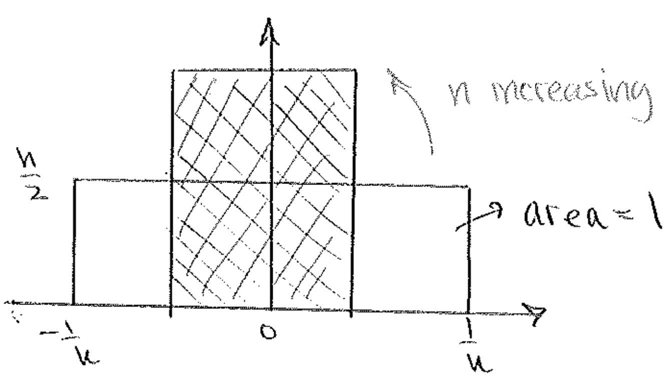
and then solve $f(x) = \delta(x - 1/2)$ in the heat equation.

- But what is this strange beast? Any classical function that satisfies (1a) is either non-integrable or has zero integral.

- Our way out is to define a sequence of functions, with $f(x) = \lim_{n\to\infty} f_n(x)$ where

$$\lim_{n\to\infty} f_n(x) = 0 \quad \forall x \neq 0 \quad \text{and} \quad \int_{-\infty}^{+\infty} f_n(x) \cdot dx = 1.$$

- Example: Consider $f_n(x) = \begin{cases} 0 & |x| > \frac{1}{n} \\ \frac{n}{2} & |x| \leq \frac{1}{n} \end{cases}$



If we return to the solution of the BVP with
$f(s) = f_n(s - 1/2)$:

$$u(x) = \underbrace{-\int_0^x (x-s) f(s)\cdot ds}_{①} + \underbrace{x \int_0^1 (1-s)\cdot f(s)\cdot ds}_{②}$$

First,
$$② = x \int_{\frac{1}{2}-\frac{1}{n}}^{\frac{1}{2}+\frac{1}{n}} (1-s)(n/2)\cdot ds = \frac{x}{2}$$

Next, ① must be split between when x is in or out
of the interval $[1/2 - \frac{1}{n}, 1/2 + \frac{1}{n}]$:

If $x < \frac{1}{2} - \frac{1}{n} \Rightarrow ① = 0$.

If $|x - 1/2| < \frac{1}{n} \Rightarrow ① = -\int_{1/2-\frac{1}{n}}^{x} (x-s) f(s)\cdot ds$

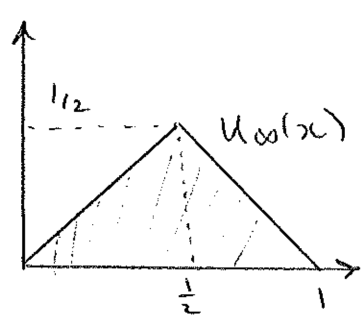
If $x > \frac{1}{2} + \frac{1}{n} \Rightarrow ① = -\int_{1/2-\frac{1}{n}}^{1/2+\frac{1}{n}} (x-s) f(s)\cdot ds.$

In total, we get:

$$u_n(x) = \begin{cases} \frac{1}{2}\cdot x & x - 1/2 < -\frac{1}{n} \\ -\frac{n x^2}{4} + \frac{n x}{4} - \frac{n}{16} + \frac{1}{4} - \frac{1}{4n} & -\frac{1}{n} \leq x - \frac{1}{2} \leq \frac{1}{n} \\ \frac{1}{2} - \frac{x}{2} & x - 1/2 > \frac{1}{n} \end{cases}$$

Pointwise limit of $u_n(x)$:

$$u_\infty(x) = \lim_{n\to\infty} u_n(x) = \begin{cases} \frac{1}{2}\cdot x & x < 1/2 \\ \frac{1}{2}\cdot(1-x) & x \geq 1/2. \end{cases}$$



In fact $u_\infty(x)$ is independent of the choice of approximating sequence so long as they satisfy (1ab)

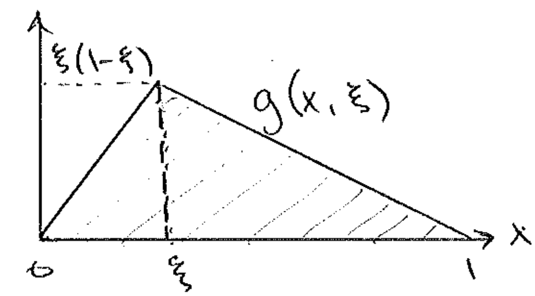
The fn. $u_\infty(x)$ is the temp. field in response to an "infinitely" concentrated heat source @ $x = 1/2$.

What about a heat source @ $x = \xi$? $(0 < \xi < 1)$

Need to solve
$$\begin{cases} g''(x,\xi) = \delta(x-\xi) \\ g(0,\xi) = 0 = g(1,\xi) \end{cases}$$

Using the same approach of sequences, we get:

$$g(x,\xi) = \begin{cases} x(1-\xi), & x \leq \xi \\ \xi(1-x), & x > \xi \end{cases}$$



But note from the explicit solution of the ODE:

$$\boxed{u(x) = \int_0^1 g(x,\xi) f(\xi) \cdot d\xi}$$

The function $g(x,\xi)$ is called the Green's Function for the ODE. Intuitively, g represents the response of the system to a point (heat) source @ $x = \xi$. The final solution is then found by weighting the source by $f(\xi)$ and summing.

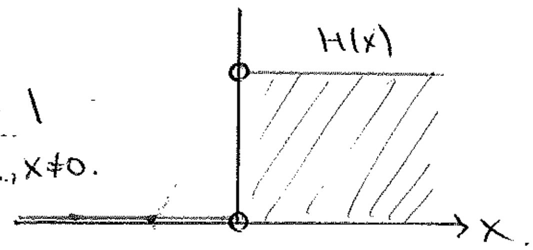
There are two important properties of δ:

Anti-derivative of δ:

$$\int_{-\infty}^{x} \delta(s)\cdot ds = \begin{cases} 0 & x < 0 \\ 1 & x > 0 \end{cases} \equiv H(x) \quad \text{(Heaviside function)}$$

justified from $\int_{-\infty}^{+\infty} \delta(x)\, dx = 1$

and $\delta(x) = 0 \ \forall x, x \neq 0$.



Sifting property:

$$\int_{-\infty}^{+\infty} \delta(x-a) f(x)\cdot dx = f(a)$$

This is also justified from approx. sequences:

$$\int_{-\infty}^{+\infty} \delta(x-a) f(x)\cdot dx = \lim_{n\to\infty} \int_{-\infty}^{+\infty} f(x)\cdot f_n(x-a)\cdot dx$$

assume limits can be manipulated in this way.

$$= \lim_{n\to\infty} \int_{a-\frac{1}{n}}^{a+\frac{1}{n}} f(x)\,\frac{n}{2}\cdot dx$$

Defining $F(x) = \int^{x} f\, ds$,

$$\int_{-\infty}^{+\infty} \delta(x-a) f(x)\, dx = \lim_{\Delta x\to 0} \frac{F(a+\Delta x) - F(a-\Delta x)}{2\Delta x} = F'(a) = f(a)$$

$$\left(\Delta x = \frac{1}{n}\right)$$

This suggests a new method for solving BVPs like $u'' = f(x)$:

(1) Find the G.F. by solving the BVP replacing $f(x)$ by $f(x) = \delta(x-\xi)$

(2) Find the solution for general f using $u(x) = \int_0^1 g(x,\xi)\cdot f(\xi)\cdot d\xi$

GFs for general linear ODEs:

Consider:
$$\begin{cases} \mathcal{L}[u] \equiv \overset{\neq 0}{a_2(x)} u'' + a_1(x) u' + a_0 u(x) = f(x) & \text{all coefs. cts.} \\ B_1[u] \equiv \alpha_{11} u(a) + \alpha_{12} u'(a) + \beta_{11} u(b) + \beta_{12} u'(b) = \gamma_1 \\ B_2[u] \equiv \alpha_{21} u(a) + \alpha_{22} u'(a) + \beta_{21} u(b) + \beta_{22} u'(b) = \gamma_2 \end{cases}$$

for $x \in [a,b]$. We assume the BVP has a unique solution.

Step 1: Determine GF $(a < \xi < b)$:
$$\begin{cases} \mathcal{L}[g(x,\xi)] = \delta(x-\xi) & \forall x \in [a,b]. \\ B_1[g] = 0 = B_2[g] \end{cases}$$

Step 2: Solve with zero B.C.
$$\bar{u}(x) = \int_a^b g(x,\xi)\cdot f(\xi)\, d\xi$$

This is a solution since
$$\mathcal{L}[\bar{u}] = \int_a^b \mathcal{L}[g(x,\xi)]\, f(\xi)\cdot d\xi = \int_a^b \delta(x-\xi)\, f(\xi)\cdot d\xi = f(x)$$

and also, $B_i[u] = \int_a^b B_i[g(x,\xi)] \cdot f(\xi) \cdot d\xi$

$$= 0.$$

note $\mathcal{L}$ is w.r.t. x and not ξ!

Step 3: Solve with non-zero B.C.s

Substitute $u(x) = v(x) + \int_a^b g(x,\xi) \cdot f(\xi) \cdot d\xi$

into the BVP. $\Longrightarrow \begin{cases} \mathcal{L}v = 0. \\ B_i[v] = \gamma_i \end{cases}$

So $v = c_1 u_1(x) + c_2 u_2(x)$ (gen. solution)

and $u = \underline{[c_1 u_1 + c_2 u_2] + \int_a^b g(x,\xi) \cdot f(\xi)\, d\xi.}$

Reformulating as a classically solvable problem

Now the question is how do we solve for $g(x,\xi)$?
In particular, we want an approach which does not use δ functions. We know:

$$\begin{cases} \mathcal{L}[g(x,\xi)] = 0 \quad \text{on } x \in (a,\xi) \cup (\xi,b) \\ \Rightarrow g(x,\xi) = \begin{cases} c_1 u_1(x) + d_1 u_2(x), & x \in (a,\xi) \\ c_2 u_1(x) + d_2 u_2(x), & x \in (\xi,b) \end{cases} \end{cases}$$

where u_1 and u_2 are lin. indep. sol'ns of $\mathcal{L}u = 0$.

We have 4 constants but only 2 more B.Cs.

$\Rightarrow$ need 2 "jump" conditions @ $x = \xi$:

When we plug $g(x,\xi)$ into the 2nd order ODE, we get:

$$\mathcal{L}[g(x,\xi)] = a_2 \frac{d^2 g}{dx^2} + \underbrace{\text{less singular terms}}_{\substack{\text{lower derivatives} \\ \text{are less singular.}}} = \delta(x-\xi)$$

If we assume that near the singularity:

$$\frac{d^2 g}{dx^2} \sim \frac{\delta(x-\xi)}{a_2(x)} \rightarrow \text{blows up.}$$

then it makes sense to posit that

$$g_{xx}(x,\xi) = \frac{\delta(x-\xi)}{a_2(x)} + \underbrace{h_{xx}(x,\xi)}_{\substack{\text{has at most a jump} \\ @\ x=\xi}}$$

anti-deriv. property $\Rightarrow$ $g_x(x,\xi) = \dfrac{H(x-\xi)}{a_2(\xi)} + h_x(x,\xi)$

$$\Rightarrow \quad g(x,\xi) = \frac{1}{a_2(\xi)} \int_a^x H(s-\xi)\,ds + h(x,\xi)$$

We conclude that g possesses the two jump conditions:

$$[g]_{x=\xi^-}^{x=\xi^+} = 0 \qquad \left[\frac{dg}{dx}\right]_{x=\xi^-}^{x=\xi^+} = \frac{1}{a_2(\xi)}$$

g's deriv. has a jump @ $x=\xi$ but g is otherwise cts.

Thus to solve
$$\begin{cases} \mathcal{L}[g(x,\xi)] = \delta(x-\xi) \\ B_i[g] = 0, \quad i = 1,2 \end{cases}$$

we can solve the classical problem:

$$\begin{cases} \mathcal{L}[g(x,\xi)] = 0. \\ B_i[g] = 0. \\ [g]_{\xi^-}^{\xi^+} = 0 \quad \text{and} \quad \left[\frac{dg}{dx}\right]_{\xi^-}^{\xi^+} = \frac{1}{a_2(\xi)} \end{cases}$$