LECTURE 20: LAPLACE'S EQN

Before we begin, we comment on solving the 1D heat eqn with inhomogeneous B.C.s:

$$\begin{cases} u_t = D u_{xx} \\ u(0,t) = \alpha \\ u(L,t) = \beta. \end{cases}$$

(NB: The difficulty with the usual approach is that if $u = T(t)X(x)$, then $X(0) = \alpha$, $X(L) = \beta$ and finding eigenvalues is difficult)

The trick is to convert the inhomog. problem to a homog. problem by subtracting the equilibrium solution. As $t \to \infty$, $u(x,t) \to u_\infty(x)$ where $u_{\infty xx} = 0$, $u_\infty(0) = \alpha$, $u_\infty(L) = \beta$. Solving gives:

$$u_\infty(x) = \alpha + \left(\frac{\beta - \alpha}{L}\right) x.$$

We can now write: $\bar{u}(x,t) = u(x,t) - u_\infty(x)$ and we must now solve:

$$\begin{cases} \bar{u}_t = D \bar{u}_{xx}. \\ \bar{u}(0,t) = 0 \\ \bar{u}(L,t) = 0 \end{cases}$$

which is straightforward using separation of variables. Neumann $\left(\frac{\partial u}{\partial x}\right)$ conditions are also handled similarly

Laplace's equation is encountered very frequently in real life applications. It is arguably the most important D.E.!

- 2D/3D heat equation: $u_t = D\underbrace{\nabla^2 u}$

  If we want equilibrium solution, then $u \to u_\infty(\vec{x})$ as $t \to \infty$ satisfies $\nabla^2 u_\infty = 0$.

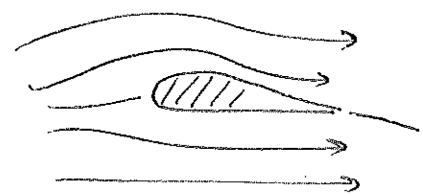
- 2D vibrations of a membrane: $u_{tt} = c^2 \nabla^2 u$

  (Laplace's eqn will be encountered through sep. of vars).

- Potential problems like in fluid mechanics

  e.g. if $\vec{u} = \nabla\phi \Rightarrow \boxed{\nabla^2\phi = 0}$

  $\vec{u}$: fluid velocity



Separation of variables: We wish to solve: $\nabla^2 u = 0 = u_{xx} + u_{yy}$

Write $u(x,y) = X(x)Y(y)$

PDE $\Rightarrow X''Y + Y''X = 0 \Rightarrow \dfrac{X''}{X} = -\dfrac{Y''}{Y} = \lambda$

So we wish to solve
$$\begin{cases} X'' - \lambda X = 0. \\ Y'' + \lambda Y = 0. \end{cases}$$

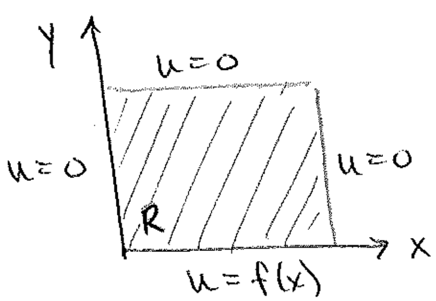
This leads to the following possibilities:

λ	$X(x)$	$Y(y)$
$\lambda=-\omega^2<0$	$\cos(\omega x), \sin(\omega x)$	$e^{-\omega y}, e^{\omega y}$
$\lambda=0$	$1, x$	$1, y$
$\lambda=\omega^2>0$	$e^{-\omega x}, e^{\omega x}$	$\cos(\omega y), \sin(\omega y)$

Which ones we use depends on B.C.s. One note we must make on the limitations of the method of separation of vars: The method is usally limited to problems where B.C.s are applied on the coordinate axes or // lines (rectangles in xy, circles and rays in polar coords, etc.). More on this when we look @ Green's Functions...

Consider solving:

$$\begin{cases} \nabla^2 u = u_{xx}+u_{yy}=0., \quad (x,y)\in R & (1) \\ u(x,0)=f(x) & (2) \\ u(x,b)=u(0,y)=u(a,y)=0. & (3)-(5) \end{cases}$$



→ note that if we can find this solution, then we can solve *any* Dirichlet problem on the rectangal by linear superposition.

First, we need $u = X(x)Y(y)$ to satisfy the (3)-(5) homogeneous conditions; so

$$X(0) = X(a) = 0$$
$$Y(b) = 0.$$

Recall $\sinh z = \frac{1}{2}(e^{z} - e^{-z})$. The condition $X(0)=0$ requires:

$$X(x) = \begin{cases} \sin(\omega x) & \lambda = -\omega^2 < 0 \\ x & \lambda = 0 \\ \sinh(\omega x) & \lambda = \omega^2 > 0. \end{cases}$$

However $X(a)=0$ rules out the second and third.
So we need. $\sin(\omega a) = 0 \Rightarrow \omega a = n\pi$, $n = 1,2,3,\ldots$

(note we have just solved eigenvalue problem
$X'' - \omega^2 X = 0$, $X(0) = X(b) = 0$)

Since $\lambda = -\omega^2 < 0$, then,

$$Y(y) = c_1 e^{-\omega y} + c_2 e^{\omega y}.$$

Need $Y(b) = 0 \Rightarrow c_1 e^{-\omega b} + c_2 e^{\omega b} = 0 \Rightarrow c_2 = -c_1 e^{-2\omega b}$

$$\therefore Y(y) = c_1 \left\{ e^{-\omega y} - e^{-2\omega b} e^{\omega y} \right\}$$
$$= \frac{c_1}{e^{\omega b}} \left\{ e^{\omega(b-y)} - e^{-\omega(b-y)} \right\} = \text{const} \times \sinh[\omega(b-y)]$$

Eigensolutions satisfying the homogeneous B.C.s:

$$u_n(x,y) = \sin\left(\frac{n\pi}{a} x\right) \sinh\left[\frac{n\pi}{a}(b-y)\right], \quad n = 1,2,3,\ldots$$

To satisfy the remaining B.C. $u(x,0) = f(x)$, we use Fourier series (note this is similar to I.C.s for the heat eqn.) Write:

$$u(x,y) = \sum_{n=1}^{\infty} c_n \cdot \sin\left(\frac{n\pi}{a} \cdot x\right) \sinh\left(\frac{n\pi}{a}(b-y)\right).$$

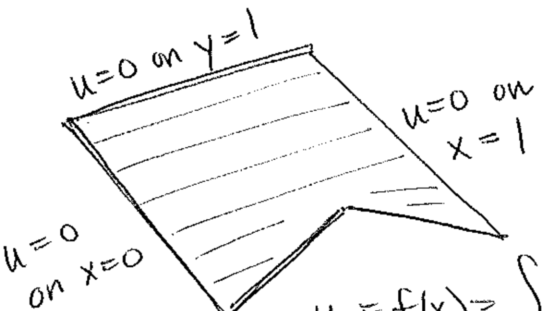
and solve:

$$f(x) = \sum_{n=1}^{\infty} c_n \sinh\left(\frac{n\pi b}{a}\right) \sin\left(\frac{n\pi}{a} \cdot x\right)$$

Applying Fourier sine coefs over $[0, a]$:

$$b_n = c_n \sinh\left(\frac{n\pi b}{a}\right) = \frac{2}{a}\int_0^a f(x) \sin\left(\frac{n\pi}{a} \cdot x\right) dx.$$

Example: Consider a membrane stretched over a wire shaped as a rectangle, but with one side bent:



$$u = f(x) = \begin{cases} x, & x \in [0, \frac{1}{2}], y = 0 \\ 1-x, & x \in [\frac{1}{2}, 1], y = 0 \end{cases}$$

We have (after computation):

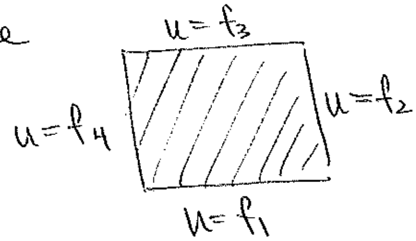
$$f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{a}\right) = \frac{4}{\pi^2} \cdot \sum_{j=0}^{\infty} (-1)^j \frac{\sin[(2j+1)\pi x]}{(2j+1)^2}$$

note $n \mapsto (2j+1)$

$$\therefore\ u(x,y) = \frac{4}{\pi^2} \cdot \sum_{j=0}^{\infty} (-1)^j \frac{\sin[(2j+1)\pi x]\sinh[(2j+1)\pi(1-y)]}{(2j+1)^2 \cdot \sinh[(2j+1)\pi]}$$

(See Matlab).

To solve Laplace's eqn for more general Dirichlet conditions like



then simply requires us to solve 4 individual problems, each one with $u_i = f_i$ on one of four boundaries and $u=0$ on the other three. By linearity, $u = \sum_1^4 u_i$ then satisfies the complete problem.

POLAR COORDINATES.

How does Laplace's eqn change in polar coordinates?

$u(x,y) \to u(r,\theta)$ where $x = r\cos\theta$, $y = r\sin\theta$, $r = \sqrt{x^2+y^2}$, $\theta = \tan^{-1}(y/x)$

By the chain rule:

$$\frac{\partial}{\partial r} = \frac{\partial x}{\partial r}\frac{\partial}{\partial x} + \frac{\partial y}{\partial r}\frac{\partial}{\partial y} = \cos\theta \frac{\partial}{\partial x} + \sin\theta \frac{\partial}{\partial y}$$

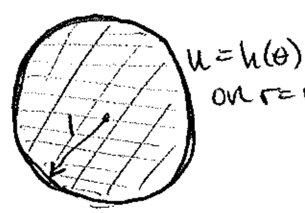
$$\frac{\partial}{\partial \theta} = -r\sin\theta \frac{\partial}{\partial x} + r\cos\theta \frac{\partial}{\partial y}.$$

So
$$\frac{\partial}{\partial x} = \cos\theta \frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial \theta}$$
$$\frac{\partial}{\partial y} = \sin\theta \frac{\partial}{\partial r} + \frac{\cos\theta}{r}\frac{\partial}{\partial \theta}$$

We can now show that

$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}$$

We wish to solve $\begin{cases} \nabla^2 u = 0. \\ u(1,\theta) = h(\theta) \end{cases}$



Separate: $u(r,\theta) = R(r)\cdot T(\theta)$

$$\nabla^2 u = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\cdot\frac{\partial^2 u}{\partial \theta^2} = R''T + \frac{1}{r}R'T + \frac{1}{r^2}T''R = 0.$$

$$\Rightarrow \frac{r^2R'' + rR'}{R} = -\frac{T''}{T} = \lambda$$

So we wish to solve $\begin{cases} T'' + \lambda T = 0. \\ r^2R'' + rR' - \lambda R = 0. \end{cases}$

Depending on the sign of λ T can be linear (if $\lambda = 0$), exp. growth/decay (if $\lambda < 0$) and cos/sin (if $\lambda > 0$)

However, the periodicity in θ requires that $T(\theta + 2\pi) = T(\theta)$, so we need

$$T(\theta) = A\cos(\lambda^{1/2}\theta) + B\sin(\lambda^{1/2}\theta)$$

Notice we need n times wavelength to be 2π for some n.

$$\Rightarrow \quad n\left(\frac{2\pi}{\lambda^{1/2}}\right) = 2\pi \quad \Rightarrow \lambda_n = n^2, \quad n = 0, 1, 2, \ldots$$

gives $T_n = 1$ for $n=0$.

This leaves us to solve

$$\underbrace{r^2 R'' + rR' - \lambda R = 0}_{\text{Cauchy-Euler equation: Let } R = r^k} \quad \text{when } \lambda \geq 0.$$

$$\Rightarrow \quad k(k-1) + k - \lambda = 0.$$

$$\Rightarrow \quad k^2 = n^2 \Rightarrow k = \pm n$$

So we have $R_n(r) = C \cdot r^n + D r^{-n}$, $n = 1, 2, 3, \ldots$

Be careful not to forget $n = 0$, $\Rightarrow r^2 R'' + rR' = 0$

$$\Rightarrow (r \cdot R')' = 0$$

$$\Rightarrow \quad R = C + D\log r$$

Now these eigenfunctions with $\log r$ are not permissible since they are unbounded as $r \to 0$.

Similarly, any eigenfunctions with r^{-n}, $n > 0$ are not permissible.

The only allowed eigensolutions are:

$$1 \text{ (const)}, \; r^n \sin(n\theta), \; r^n \cos(n\theta).$$

So we form the general series solution:

$$u(r,\theta) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left\{ a_n \cos(n\theta) \cdot r^n + b_n \sin(n\theta) \cdot r^n \right\}$$

note shift to $\frac{1}{2}$ factor.

To apply the Dirichlet condition $u(1,\theta) = h\theta$, we can use the Fourier $[-\pi, \pi]$ coefs:

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} h(\theta) \cdot \cos(n\theta) \cdot d\theta \qquad b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} h(\theta) \sin(n\theta) \cdot d\theta .$$

__Example__: Consider the solution of $\nabla^2 u = 0$ on $0 \le r \le 1$ and $-\pi \le \theta \le \pi$ with $u(1,\theta) = h(\theta) = \theta$

The Fourier series for $h(\theta)$ is:

$$h(\theta) = \theta = 2 \left\{ \sin\theta - \frac{\sin 2\theta}{2} + \frac{\sin 3\theta}{3} + \cdots \right\}.$$

$$u(r,\theta) = 2 \left\{ r\sin\theta - r^2 \frac{\sin 2\theta}{2} + r^3 \frac{\sin 3\theta}{3} + \cdots \right\}.$$

so $a_n = 0, n \geq 0$

$b_n = \frac{2}{n}(-1)^{n+1}, n \geq 1$

(see Matlab)