# LECTURE 19: SEPARATION OF VARS. AND FOURIER SERIES

Last time, we looked at the heat distribution in a 1D bar using the heat eqn: $\frac{\partial u}{\partial t} = D \frac{\partial^2 u}{\partial x^2}$.

To review: $u(x,t) = T(t) \cdot X(x)$

$$\text{PDE} \Rightarrow T'(t) \cdot X(x) = D \cdot T(t) \cdot X''(x)$$

$$\Rightarrow \frac{T'(t)}{D \cdot T(t)} = D \cdot \frac{X''(x)}{X(x)} = -\lambda^2 < 0$$

you will be asked in your PS to justify why we take this as negative.

Solving for $T(t)$:

$$\Rightarrow T(t) = \text{Const} \times e^{-\lambda^2 D \cdot t}$$

Solving for $X(x)$ with insulated ends: $(T(0,t) = 0 = T(L,t))$

$$\Rightarrow X(x) = \text{Const} \times \sin(\lambda x) \qquad \lambda = \frac{n\pi}{L},\ n \in \mathbb{Z}$$

The eigensolutions are:

$$u_n(x,t) = B_n T_n(t) \cdot X_n(x)$$

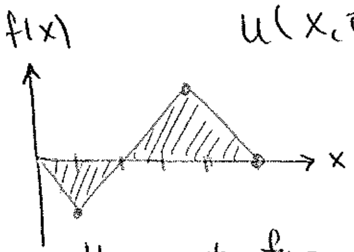
$$= B_n e^{-\lambda_n^2 D t} \sin(\lambda_n x)$$

with $\lambda_n = \frac{n\pi}{L},\ n = 1, 2, \ldots$

Given an initial condition, $u(x,0) = f(x)$, we need to solve for the Fourier sine series:

$$f(x) = \sum_{n=1}^{\infty} B_n \cdot \sin(\lambda_n x) \xrightarrow[\text{Fourier coefs.}]{} B_n = \frac{2}{L} \int_0^L f(x) \sin\left(\frac{n\pi x}{L}\right) \cdot dx.$$

Instead of using $f(x) = \text{const}$ (as for a constantly initially heated bar) let's use the initial heat:




$$u(x,0) = f(x) = \begin{cases} -x & 0 \le x \le \frac{1}{5} \\ x - \frac{2}{5} & \frac{1}{5} \le x \le \frac{7}{10} \\ 1-x. & \frac{7}{10} \le x \le 1 \end{cases}$$

We get for the first few coefs. (for $L=1$)

| $n$ | $b_n$ |
|---|---|
| 1 | 0.090 |
| 2 | −0.193 |
| 3 | −0.029 |
| 4 | 0.000 |
| 5 | −0.016 |
| ⋮ | ⋮ |

$$u(x,t) = \sum_{n=1}^{\infty} B_n \cdot e^{-\lambda_n^2 D t} \sin\left(\frac{n\pi x}{L}\right)$$

$$\approx (0.090)\, e^{-D\pi^2 t} \sin(\pi x)$$
$$+ (-0.193)\, e^{-2^2\pi^2 D t} \sin(2\pi x)$$
$$+ (-0.029)\, e^{-3^2\pi^2 D t} \sin(3\pi x)$$
$$+ \dots$$

See Matlab for a plot of the solution at different times. Note:

- Corners in the initial profile are immediately smoothed
- As $t \to \infty$, the solution decays very quickly (at the dominant rate $e^{-D\pi^2 t}$) to the equilibrium solution $u \equiv 0$.
- As $t \to \infty$, the higher Fourier modes (eigensolutions) disappear rapidly, leaving mostly the lower modes.

Smoothing and long-term behaviour:

If $f(x)$ is integrable (e.g. piecewise cts.) then the Fourier coefs. are uniformly bounded:

$$|B_n| \le \frac{2}{L}\int_0^L \left| f(x)\sin\left(\frac{n\pi x}{L}\right)\right| dx$$
$$\le \frac{2}{L}\int_0^L |f(x)|\,dx \equiv M.$$

Then the Fourier modes:

$$|B_n \cdot T_n(t)\cdot X_n(x)| = \left| b_n \cdot e^{-\frac{Dn^2\pi^2}{L^2}\cdot t}\cdot \sin\left(\frac{n\pi x}{L}\right)\right|$$
$$\le M e^{-\frac{Dn^2\pi^2}{L^2}t}$$

- As soon as $t>0$, all the high frequency modes, with $n \gg 0$ will be very small. We only "see" the low frequency modes.
- As $t\to\infty$, all modes decay to zero. The small-scale temperature variations quickly disappear from diffusion (spreading) of heat energy. The last term to disappear is:

$$u(x,t) \approx B_1 e^{-\frac{D\pi^2 t}{L^2}} \sin\left(\frac{\pi x}{L}\right) \qquad (\text{so long as } B_1 \ne 0).$$

- Because of the fact that $u(x,t)$ converges to $u \equiv 0$ exp. fast, then regardless of the initial profile, the Fourier sine series converges to an infinitely differentiable function of $x$ $\forall\, t>0$.

The following can be proved:

Theorem: If $u(x,t)$ is a solution of the heat equation with piecewise cts. I.C. $u(x,0) = f(x)$, then $\forall\ t > 0$, the solution $u(x,t)$ is an infinitely differential function of $x$.

- What about reversing time, $t \mapsto -t$ ?

$$\Rightarrow \frac{\partial u}{\partial t} = -D \frac{\partial^2 u}{\partial x^2}$$

backwards heat equation.

The problem is now ill-posed (for most initial data, the solution is not defined for $t > 0$). Note that $\exists$ many applications of solving the backwards heat equation by "regularizing" it (making it well-posed).

e.g. Sharpening images, forensics (determining the time of death), reconstructing terrain data from seismic data, etc.

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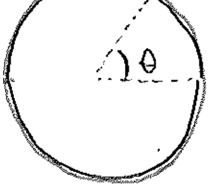
Example: (Heated ring)

Consider a heated ring of length $-\pi \leq \theta \leq \pi$ and $D = 1$




$$\left.\begin{array}{ll} \dfrac{\partial u}{\partial t} = \dfrac{\partial^2 u}{\partial \theta^2} & u = u(\theta,t) \\ u(-\pi,t) = u(\pi,t) & \text{[Periodic B.C.s]} \\ \dfrac{\partial u}{\partial x}(-\pi,t) = \dfrac{\partial u}{\partial x}(\pi,t) & \end{array}\right.$$

$$u(\theta,0) = f(\theta) \qquad -\pi \leq \theta \leq \pi$$

Separating $u = T(t)\cdot F(\theta) \Rightarrow \frac{T'}{T} = \frac{F''}{F} = -\lambda^2 < 0.$

Solving $T(t)$: $T(t) = \text{const} \times e^{-\lambda^2 t}$

Solving $F(\theta) \Rightarrow F'' + \lambda^2 F = 0$

$$\Rightarrow \quad F = A\cos(\lambda\theta) + B\sin(\lambda\theta)$$

B.C.s $\quad u(-\pi, t) = u(\pi, t) \Rightarrow F(-\pi) = F(\pi) \quad (1)$

$\quad\quad u_x(-\pi, t) = u_x(\pi, t) \Rightarrow F'(-\pi) = F'(\pi) \quad (2)$

$(1) \Rightarrow A\cos(\lambda\pi) + B\sin(\lambda\pi) = A\cos(\lambda\pi) - B\sin(\lambda\pi)$

$\Rightarrow \quad 2B\sin(\lambda\pi) = 0. \quad (3)$

$(2) \Rightarrow -\lambda A\sin(\lambda\pi) + B\lambda\cos(\lambda\pi) = A\lambda\sin(\lambda\pi) + B\lambda\cos(\lambda\pi)$

$\Rightarrow \quad 2A\lambda\sin(\lambda\pi) = 0. \quad (4)$

$(3) + (4) \Rightarrow$ need $\lambda\pi = 0 \Rightarrow \lambda = \frac{n\cancel{\pi}}{\cancel{\pi}}, \ n \in \mathbb{Z}.$

The eigensolutions are:

$$u_n(\theta, t) = e^{-\lambda_n^2 t}\left\{A_n\cos(\lambda_n\theta) + B_n\sin(\lambda_n\theta)\right\}.$$

$$\lambda_n = n, \quad n = 0, 1, 2, 3, \ldots$$

note we've absorbed $n = -\ldots, -2, -1$ into the coefs. Also, we must include $n = 0$ since $\cos(\lambda_0\theta) = 1$.

But now need to impose I.C.s.

$f(x) = u(x, 0).$

$\frac{1}{2}$ factor for convenience.

We write the solution as $u(x, t) = \frac{A_0}{2} + \sum_{n=1}^{\infty} e^{-\lambda_n^2 t}\left\{A_n\cos(\lambda_n\theta) + B_n\sin(\lambda_n\theta)\right\}$

By the $[-\pi, \pi]$ Fourier coefficients:

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(n\theta)\cdot d\theta \qquad n \geq 0.$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(n\theta)\cdot d\theta.$$

Note that as $t \to \infty$, the high modes quickly disappear, and as $t \to \infty$,

$$u(x,t) \to \frac{A_0}{2} = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\cdot dx$$

From your PS, you derived this exact result:

$$\begin{matrix}\text{Equilibrium total} \\ \text{heat}\end{matrix} = \underbrace{(2\pi)}_{\text{area}}\ \underbrace{u_\infty(x)}_{\infty \text{ temp.}} = \underbrace{\int_{-\pi}^{\pi} f(x)\, dx}_{\text{initial heat}}$$

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SEPARATION OF VARIABLES can also be used to solve the wave equation

$$\frac{\partial^2 u}{\partial t^2} = c^2 \cdot \frac{\partial^2 u}{\partial x^2} \qquad \text{with} \quad \begin{matrix} u(0,t) = 0 \\ u(L,t) = 0 \end{matrix}.$$

over finite domains. (Previously we used the method of characteristics and d'Alembert's formula). Write:

$$u(x,t) = T(t)\cdot X(x)$$

$$\Rightarrow T''\cdot X = c^2 \cdot X'' T \Rightarrow \frac{T''}{T} = \frac{c^2 X''}{X} = -\lambda^2 < 0.$$

(why < 0? check T(t))

Solving T(t):

$$T'' = -\lambda^2 T \Rightarrow T(t) = A\cos(\lambda t) + B\sin(\lambda t).$$

note the oscillatory $T(t)$ is what we expect for the wave equation. Solving for $X(x)$:

$$X''(x) + \frac{\lambda^2}{c^2} \cdot X(x) = 0$$

$$\Rightarrow X(x) = C\cos\left(\frac{\lambda x}{c}\right) + D\sin\left(\frac{\lambda x}{c}\right)$$

B.C.s: $u(0,t) = T(t)\cdot X(0) = 0 \Rightarrow X(0) = 0.$
$u(L,t) = T(t)\cdot X(L) = 0 \Rightarrow X(L) = 0.$

$$\Rightarrow X(x) = D\sin\left(\frac{\lambda x}{c}\right) \quad \text{where} \quad \frac{\lambda L}{c} = n\pi, \; n \in \mathbb{Z}$$

$$\Rightarrow \lambda = \frac{cn\pi}{L}, \; n \in \mathbb{Z}$$

Thus, the eigensolutions are:

$$u_n(x,t) = A_n \cos(\lambda_n t)\sin(\lambda_n x/c) + B_n \sin(\lambda_n t)\sin(\lambda_n x/c)$$

$$\lambda_n = \frac{cn\pi}{L}, \quad \underbrace{n = 1, 2, 3, \ldots}$$

note we drop $n = \ldots, -2, -1, 0$.

Given an I.C.s:

$$\begin{cases} u(x,0) = f(x) \\ \frac{\partial u}{\partial t}(x,0) = g(x) \end{cases}$$

We express $u$ as Fourier series:

$$u(x,t) = \sum_{n=1}^{\infty} \left\{ A_n \cos(\lambda_n t)\sin\left(\frac{\lambda_n x}{c}\right) + B_n \sin(\lambda_n t)\sin\left(\frac{\lambda_n x}{c}\right) \right\}$$

and we solve:

$$f(x) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{\lambda_n x}{c}\right) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{L}\right)$$

and $g(x) = \sum_{n=1}^{\infty} B_n \cdot \lambda_n \sin\left(\frac{\lambda_n x}{c}\right) = \sum_{n=1}^{\infty} B_n \left(\frac{cn\pi}{L}\right) \sin\left(\frac{n\pi x}{L}\right)$

Using the Fourier sine coefficients over $[0, L]$ gives:

$$A_n = \frac{2}{L} \int_0^L f(x) \sin\left(\frac{n\pi x}{L}\right) \cdot dx$$

$$B_n = \frac{2}{n\pi c} \int_0^L g(x) \sin\left(\frac{n\pi x}{L}\right) \cdot dx$$

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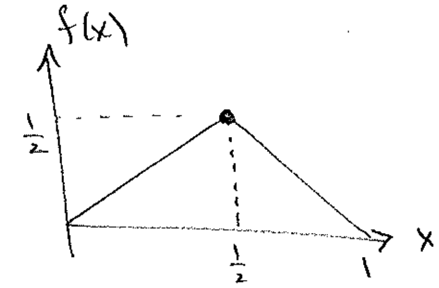
<u>Example</u> : (Plucked finite string)

$$\begin{cases} u_{tt} = u_{xx}. \\ u(x,0) = f(x) = \begin{cases} x & 0 \le x \le \frac{1}{2} \\ 1-x & \frac{1}{2} < x \le 1 \end{cases} \\ u_t(x,0) = 0 \\ u(0,t) = u(1,t) = 0. \end{cases}$$




Here, $\lambda_n = n\pi$, $n = 1, 2, 3, \ldots$

$$u(x,t) = \sum_{n=1}^{\infty} \left\{ A_n \cos(n\pi t) \sin(n\pi x) + B_n \sin(n\pi t) \sin(n\pi x) \right\}.$$

note $B_n = 0$ since $u_t(x,0) = g(x) \equiv 0$.

By computing $A_n$, we can show that

$$u(x,t) = \frac{4}{\pi^2} \sum_{k=0}^{\infty} (-1)^k \frac{\cos[(2k+1)\pi t] \sin[(2k+1)\pi x]}{(2k+1)^2}$$

(note $k = \frac{n}{2}$)

(See Matlab for plot)