LECTURE 18: FOURIER SERIES

There are three important second-order linear PDEs:

(1) Wave equation: $u_{tt} = c^2 u_{xx}$

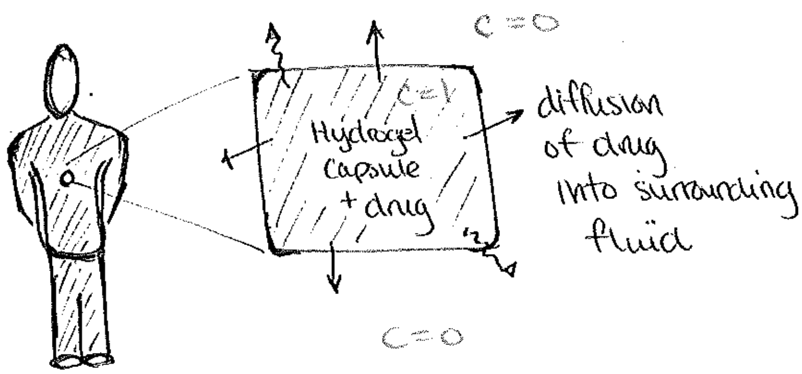
(2) Heat equation: $u_t = \gamma u_{xx}$.

(3) Laplace's Equation: $u_{xx} + u_{yy} = 0$.

We have solved (1) using characteristics. Now, we look to develop a technique for solving (2), and in particular, for dealing with solutions on finite intervals.

To keep our eye on the prize, let's review some scenarios where the heat equation $u_t = \gamma u_{xx}$ (1D) or $u_t = \gamma \nabla^2 u$ (2D or 3D) arises:

DRUG DELIVERY (STATIC)

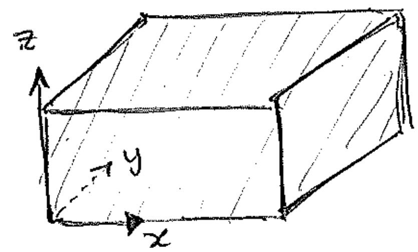


$$\frac{\partial c}{\partial t} = \nabla \cdot (D \nabla c)$$

c = concentration (mol/cm³)

D = diffusion coefficient of the drug in the capsule (cm²/s)

BAKING CAKES



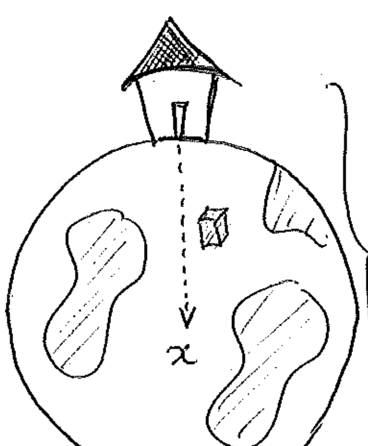
T_b = constant

- rectangular cake
- constant diffusion constant (depending on cake materials) $[cm^2/s]$
- Temperature $T(x,y,z,t)$ [Kelvins].
- Cake starts at some initial temperature $T(x,y,z,0) = T_0$.

$$\begin{cases} \dfrac{\partial T}{\partial t} = D\nabla^2 T = D(T_{xx} + T_{yy} + T_{zz}) \\ T(x,y,z,0) = T_0. \\ \qquad T = T_b \quad \text{on all boundaries} \end{cases}$$

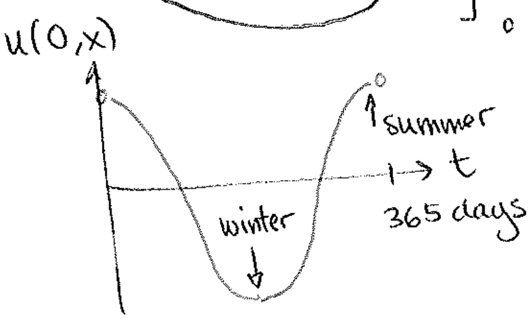
→ How long will it take the center of the cake to reach a desired temperature?

→ How do we optimize the cake (shape, materials, etc.) to achieve best results.



- How deep do you build a root cellar? (so that food is kept @ constant temperature?)
- $u(t,x)$ = deviation of temperature from annual mean
- Temperature on surface fluctuates:

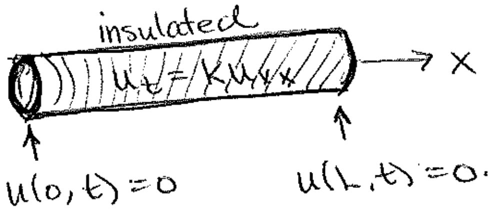
$$u(t,0) = a\cos\omega t, \quad \omega = \frac{2\pi}{365 \text{ days}}$$



- Temperature deep below does not vary: $u(t,x) \to 0$ as $x \to \infty$

Solve:
$$\begin{cases} u_t = \gamma u_{xx}, & 0 < x < \infty \\ u(t,0) = a\cos(\omega t) \\ u \to 0 \text{ as } x \to +\infty \end{cases}$$

These problems are a little advanced, so we start with a "one-dimensional cake":

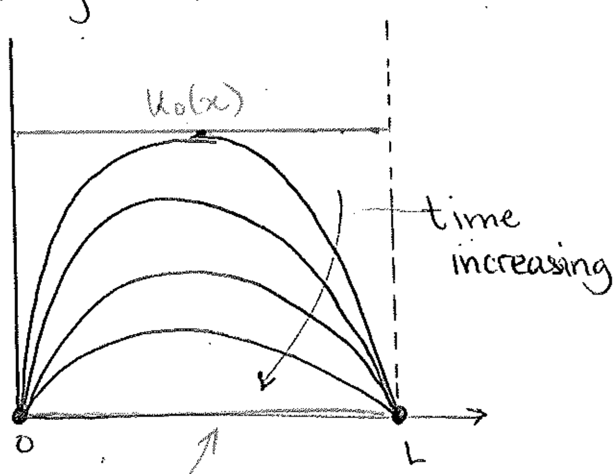


I.C. $\underbrace{u(x,0) = u_0(x)}_{\text{initial temperature}}$

Key idea: Recall from our study of differential equations that solutions occur as, $y = c_1 e^{r_1 x} + c_2 e^{r_2 x} + \cdots,$ i.e. solution space is represented using a finite basis of eigenfunctions (e^{rx}) and eigenvalues (r).

In the study of PDEs, the same idea follows, i.e. build up the sol'n as a combination of pieces, but now the # of pieces may be infinite.

What can we expect?



as $t \to \infty$, does it approach $u = 0$?

The trick is to assume that solutions are separable:

$$u(x,t) = T(t)\cdot X(x)$$

$$\text{PDE} \Rightarrow \underbrace{T'(t)\cdot X(x)}_{u_t} = k\underbrace{T(t)X''(x)}_{u_{xx}}$$

$$\Rightarrow \underbrace{\frac{T'(t)}{kT(t)}}_{\text{fn. of } t} = \underbrace{\frac{X''(x)}{X(x)}}_{\text{fn. of } x.}$$

Since we have a fn. of t on the left and a fn. of x on the right, the only way this can happen is if LHS = RHS = const.

Let us assume $\dfrac{T'(t)}{kT(t)} = \dfrac{X''(x)}{X(x)} = \underbrace{-\lambda^2}$, $\lambda > 0$ (*)

why $-\lambda^2$? Stay tuned...

$$(*) \Rightarrow T'(t) = -k\lambda T(t) \Rightarrow \boxed{T(t) = Ae^{-\lambda^2 kt}}$$

notice that had $-\lambda^2 < 0$, then $T(t)$ = exp. growth as $t \to \infty$. This does not make physical sense (we can prove $-\lambda^2 > 0$ by contradiction).

$$(*) \Rightarrow X'' + \lambda^2 X = 0 \Rightarrow X(x) = B\cos(\lambda x) + C\sin(\lambda x)$$

B.C.s : $u(0,t) = 0 \Rightarrow X(0) = 0 \Rightarrow B = 0.$

$u(L,t) = 0 \Rightarrow X(L) = 0 \Rightarrow C\sin(\lambda L) = 0.$

Since $C \neq 0$, we need $\sin(\lambda L) = 0$ or

$$\underbrace{\lambda = \lambda_n = \frac{n\pi}{L} \quad \text{for } n \in \mathbb{Z}.}_{\text{These are the eigenvalues}}$$

- note $n=0$ gives trivial solution.
- also, $n \in \mathbb{Z}^-$ gives negative coefficients, so they can be ignored.
- We conclude that the eigenvalues and eigenfunctions are

$$X_n(x) = \sin(\lambda_n \cdot x) \quad \text{where } \lambda_n = \frac{n\pi}{L}, \quad n = 1, 2, 3, \ldots$$

- The eigensolutions are:

$$\underbrace{u_n(x,t) = e^{-\lambda_n^2 k t} \sin(\lambda_n x) \quad , n = 1, 2, 3, \ldots}_{\text{But these solutions do not satisfy the initial condition } u(x,0) = u_0(x).}$$

- Since linear combinations of solutions are still solutions, then

$$u(x,t) = \sum_{n=1}^{\infty} A_n e^{-\lambda_n^2 k t} \sin(\lambda_n \cdot x) \quad \text{is a solution.}$$

In order to satisfy the I.C. we need.

$$u_0(x) = \sum_{n=1}^{\infty} A_n \cdot \sin\left(\frac{n\pi x}{L}\right) \qquad (*)$$

(*) seems to suggest that $u_0(x)$ must take a very special form (ie. $u_0(x)$ must be a sum of sines). What Fourier realised in 1827 (which was <u>very</u> controversial at the time) is that practically any (!) function $u_0(x)$ defined on $[0,L]$ can be expanded into an infinite series of sines & cosines. In music, this is the idea that any sound can be broken-up into a sum of "pure tones"! This is the theory of FOURIER SERIES.

But how do we find A_n in (*)?

Recall in linear algebra, given orthonormal basis $\{\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_n\}$ and $\vec{v} = c_1\vec{u}_1 + c_2\vec{u}_2 + \cdots + c_n\vec{u}_n$, to find c_k, multiply (dot) both sides by $\vec{u}_k$:

$$\vec{v} \circ \vec{u}_k = c_1 \cancel{\vec{u}_1 \circ \vec{u}_k} + \cdots + c_k(\vec{u}_k \circ \vec{u}_k) + 0 + 0 \cdots$$

$$\therefore \quad c_k = \frac{\vec{v} \circ \vec{u}_k}{\|\vec{u}_k\|} .$$ Same idea here.

In fact, $\{\sin(\lambda_n x)\}$ forms an orthogonal, infinite-dimensional basis. Given two functions, the "dot product" is now $\int_0^L f(x)\cdot g(x)\cdot dx$.

So we solve $u_0(x) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{L}\right)$ by multiplying by $\sin\left(\frac{m\pi x}{L}\right)$ and integrating.

THEOREM: $$\begin{cases} \int_0^L \sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{m\pi x}{L}\right)\cdot dx = 0 & \text{if } m \neq n. \\ \int_0^L \ldots \longrightarrow \cdot dx = \frac{L}{2} & \text{if } m = n. \end{cases}$$

Pf: Since $\sin a \cdot \sin b = \frac{1}{2}\cos(a-b) - \frac{1}{2}\cos(a+b)$.

Thus: $$\int_0^L \sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{m\pi x}{L}\right)\cdot dx = \frac{1}{2}\int_0^L \cos\left[\frac{(n-m)\pi x}{L}\right]\cdot dx - \frac{1}{2}\int_0^L \cos\left[\frac{(n+m)\pi x}{L}\right]\cdot dx$$

$$= \frac{1}{2}\left[\frac{L}{(n-m)\pi}\right]\sin\left[\frac{(n-m)\pi x}{L}\right]\Bigg|_0^L - \{\text{same with } (n+m)\}$$

$= 0$ if $n \neq m$ since only terms $\sin(k\pi x)$ occur.

$\int_0^L \sin^2\left(\frac{n\pi x}{L}\right) dx = \frac{L}{2}$ by direct integration ($\sin^2 a = \frac{1}{2}\{1-\cos(2a)\}$).

$\boxtimes$

We return to the statement $u_0(x) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{L}\right)$

and form the inner product with $\sin\left(\frac{n\pi x}{L}\right)$

$$\Rightarrow \int_0^L u_0(x) \sin\left(\frac{n\pi x}{L}\right) \cdot dx = A_n \cdot \left(\frac{L}{2}\right)$$

$$\Rightarrow \boxed{A_n = \frac{2}{L} \int_0^L u_0(x) \sin\left(\frac{n\pi x}{L}\right) \cdot dx}$$

which are the Fourier sine coefficients.

Consider: $\begin{cases} u_t = k u_{xx} \\ u(0,t) = u(L,t) = 0 \\ u(x,0) = u_0(x) \equiv 1 \end{cases}$

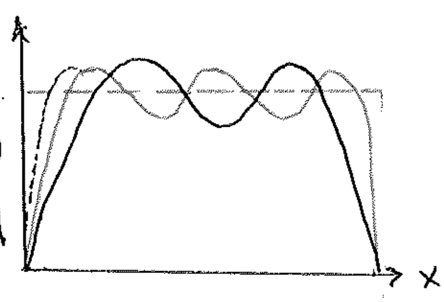
$$\text{Then } A_n = \frac{2}{L} \int_0^L \sin\left(\frac{n\pi x}{L}\right) \cdot dx = \frac{-2}{n\pi} \cos\left(\frac{n\pi x}{L}\right)\Bigg|_0^L$$

$$= \frac{2}{n\pi}\left\{1 - \underset{\uparrow \atop \cos(n\pi)}{(-1)^n}\right\}$$

$$\therefore A_n = \begin{cases} 0 & \text{if } n \text{ even} \\ \frac{4}{n\pi} & \text{if } n \text{ odd.} \end{cases}$$

$$u_0(x) = 1 = \frac{4}{\pi}\left\{\sin\left(\frac{\pi x}{L}\right) + \frac{1}{3}\sin\left(\frac{3\pi x}{L}\right) + \frac{1}{5}\sin\left(\frac{5\pi x}{L}\right) + \dots\right\}$$

notice pointwise (slow) convergence @ $x = 0, L$

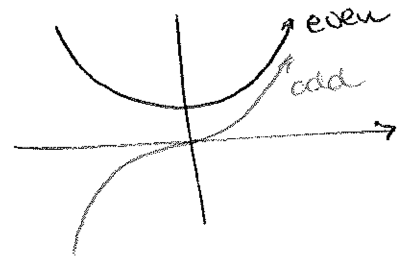


For a plot of the solution $u(x;t) = \sum_{n=1}^{N} A_n e^{-\lambda_n^2 kt} \sin(\lambda_n x)$,

see Matlab sheet.

Before we go further, we shall spend some time reviewing (developing) the theory of Fourier series.

Recall that an even function satisfies $f(-x) = f(x)$ and an odd function satisfies $f(-x) = -f(x)$



Consider the example of

$f(x) = x - x^3$. on $0 \le x \le L = 1$

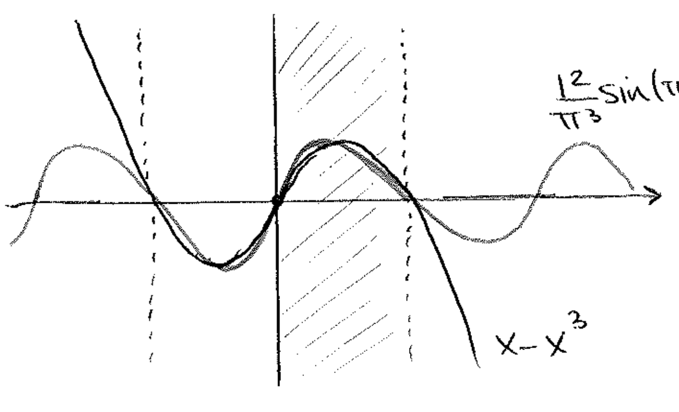
Suppose we assume that $\exists$ a_n s.t.

$$f(x) = \sum_{n=1}^{\infty} a_n \sin(n\pi x) \Rightarrow a_n = 2\int_0^1 (x - x^3)\sin(n\pi x) \cdot dx$$

$$= -\frac{12(-1)^n}{(n\pi)^3}$$

(after algebra).

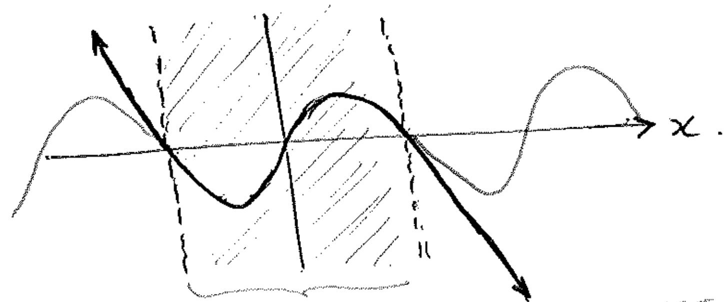
We get the following results using $f(x) \approx \frac{12}{\pi^3}\sin(\pi x)$



- The Fourier series is a good approximation throughout $0 \le x \le 1$.

- But note that outside the interval $0 \le x \le 1$, Fourier series and function do not agree. (after all, how could it, since a_n only knows about $f(x)$ on $0 \le x \le L$). By virtue of the oddness of $f(x)$, the fit is equally good on $-1 \le x \le 0$.

- Finally, we see that $\sum a_n \sin(n\pi x)$, outside $|x| > L = 1$, approximates the "periodic extension" of $x - x^3$



this piece is copied over $-L \le x \le 2L$, $-2L \le x \le -L$, etc.

Consider now the Fourier sine series for $f(x) = x$ over $0 \le x \le 1$.

$$\Rightarrow f(x) = \sum_{n=1}^{\infty} a_n \sin(n\pi x) \quad \text{and} \quad a_n = 2\int_0^1 x \cdot \sin(n\pi x)\, dx.$$

$$= \frac{-2\,(-1)^n}{(n\pi)}$$

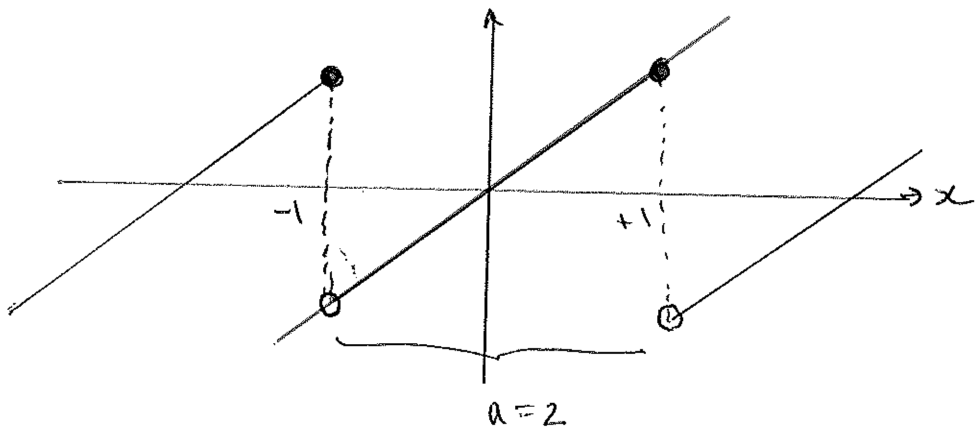
(This one is difficult to plot well...)
see Matlab sheet.

- The key is that the Fourier series, over the entire $\mathbb{R}$, produces an approximation not to $f(x) = x$, but to an extension of $f(x) = x$. We can view $f(x) = x$, defined on $(-1, 1]$

The periodic extension is defined as follows:

given any $x \in \mathbb{R}$, find the unique integer m s.t.

$x - 2m \in (-1, 1]$. Then set $F(x) = f(x - ma)$



- What about $x = \pm 1, \pm 2, \pm 3, \dots$?

At points of discontinuity, the Fourier series converges to the mean value of the jump:

$$\underset{\text{series}}{f(x)} = \frac{f(x^+) - f(x^-)}{2} = \frac{[f]^+_-}{2}$$

→ away from discontinuities, the Fourier series converges to the exact value of $f(x)$.

- Moreover, @ a point of discontinuity the partial sums always overshoot the limiting values (by about 9%).

The overshoot does not tend to zero as $n \to \infty$, but the width of the overshooting region does tend to zero. This is the Gibbs Phenomenon

FOURIER SERIES

Let $f(x)$ be an arbitrary function defined on $(-L, L]$.

The infinite series,

$$\frac{1}{2}a_0 + \sum_{n=1}^{\infty} \left\{ a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right) \right\}$$

is called the Fourier series for $f(x)$ if the coefficients a_n and b_n are given by

$$\left.\begin{aligned} a_n &= \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right) \cdot dx \\ b_n &= \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right) \cdot dx \end{aligned}\right\} \text{Fourier coefficients.}$$

This series converges to the $2L$-periodic extension of $f(x)$, defined by.

$$F(x) = f(x) \quad \text{for } x \in (-L, L].$$
$$F(x) = f(x+2L) \quad \forall\, x.$$

THEOREM :

(i) If $F(x)$ and $f(x)$ are piecewise continuous, the Fourier series converges pointwise to. $F(x)$ at each point where $F(x)$ is continuous. At each point where $F(x)$ has a jump discontinuity then

$$\frac{1}{2}\left\{F(x^+) - F(x^-)\right\} = \frac{1}{2}a_0 + \sum_{n=1}^{\infty}\left\{a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right)\right\},$$

so the Fourier series converges to the mean value of the jump.

(ii) If F is continuous and F' is sectionally (piecewise) continuous then the Fourier series converges uniformly to $F(x)$.

(iii) If F is C^p and $F^{(p+1)}(x)$ is piecewise continuous, the series obtained by differentiating the Fourier Series termwise j-times, $j = 0, 1, \cdots, p$ converges uniformly to $F^{(j)}(x)$.