LECTURE 17

THE WAVE EQN.

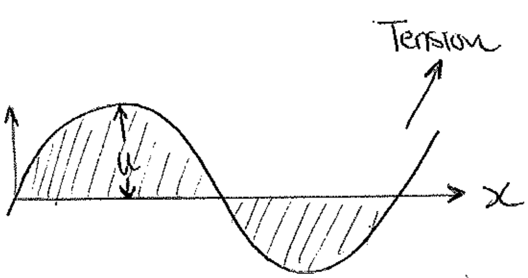
We now have a good understanding of first-order transport equations like $u_t + c(u) \cdot u_x = 0$. In fact the theory of characteristics can be used to solve first-order quasi-linear equations:

$$a(x,y,u) \cdot \frac{\partial u}{\partial x} + b(x,y,u) \frac{\partial u}{\partial y} = c(x,y,u)$$

[See Howison, p. 81].

We move on to look at second-order PDEs. Recall the wave equation:

$$\boxed{\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2},}$$ WAVE EQUATION



where c = wave speed > 0, and this equation was derived using Newton's Laws applied to a string with tension.

Physically, we expect the pair of conditions

$$\begin{cases} u(0,x) = f(x) & \rightarrow \text{initial position} \\ u_t(0,x) = g(x) & \rightarrow \text{initial velocity.} \end{cases}$$

We only consider (for now) solutions on $-\infty < x < \infty$.
Solutions on bounded intervals require other techniques.

If $\mathcal{L} = \partial_t^2 - c^2 \partial_x^2$, then the PDE is,

$$\mathcal{L}u = \left(\frac{\partial^2}{\partial t^2} - c^2 \frac{\partial^2}{\partial x^2}\right) u = 0.$$

Assuming equality of mixed partials,

$$\mathcal{L} = \underbrace{\left(\frac{\partial}{\partial t} - c\frac{\partial}{\partial x}\right)}_{\oplus} \underbrace{\left(\frac{\partial}{\partial t} + c\frac{\partial}{\partial x}\right)}_{\circledast}$$

$\circledast$ is a first-order transport equation. But we know that the solution to $u_t + c u_x = 0$ is

$$u(t,x) = p(\xi) = p(x - ct)$$

which is a wave travelling rightwards at speed c,
As long as $p(\xi)$ is twice differentiable, then if $u = p(\xi)$,
$\mathcal{L}u = \left(\frac{\partial}{\partial t} - c\frac{\partial}{\partial x}\right)\underbrace{\left(u_t + c u_x\right)}_{=0} = 0$, so $p(\xi)$ must also be a solution of the wave equation. Similarly, if we solve the "backwards" transport equation:
$u_t - c u_x = 0$, then we find

$$u(t,x) = q(\eta) = q(x + ct)$$

is also a solution of $\mathcal{L}u = 0$.

Thus, the wave equation has $\underline{2}$ characteristic variables, $\xi = x - ct$ and $\eta = x + ct$ and admits both left and right travelling waves.

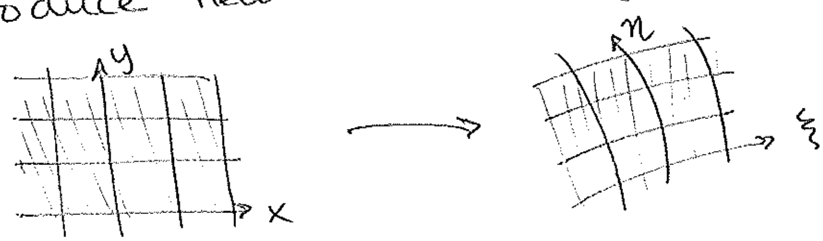
In fact, by linearity $p(\xi) + q(\eta)$ is also a solution

THEOREM: Every solution to the wave equation $u_{tt} - c^2 u_{xx} = 0$ can be written

$$u(t,x) = p(\xi) + q(\eta) = p(x-ct) + q(x+ct),$$

i.e. a superposition of left & right traveling waves.

PROOF: Introduce new coordinate system



$$u(t,x) = v(x-ct, x+ct) = v(\xi, \eta) \quad \text{with} \begin{cases} \xi = x - ct \\ \eta = x + ct \end{cases}$$

Note: $\partial_t = \frac{\partial \xi}{\partial t} \frac{\partial}{\partial \xi} + \frac{\partial \eta}{\partial t} \frac{\partial}{\partial \eta} = c\left(-\partial_\xi + \partial_\eta\right)$

$$\partial_x = \frac{\partial \xi}{\partial x} \frac{\partial}{\partial \xi} + \frac{\partial \eta}{\partial x} \frac{\partial}{\partial \eta} = \left(\partial_\xi + \partial_\eta\right)$$

$$\Rightarrow \partial_{tt} = c^2\left(-\partial_\xi + \partial_\eta\right)\left(-\partial_\xi + \partial_\eta\right) = c^2\left(\partial_{\xi\xi} - 2\partial_{\xi\eta} + \partial_{\eta\eta}\right)$$

$$\partial_{xx} = \left(\partial_\xi + \partial_\eta\right)\left(\partial_\xi + \partial_\eta\right) = \left(\partial_{\xi\xi} + 2\partial_{\xi\eta} + \partial_{\eta\eta}\right)$$

Thus, $\mathcal{L}u = \frac{\partial^2 u}{\partial t^2} - c^2\frac{\partial^2 u}{\partial x^2} = c^2\left(\partial_{\xi\xi} - 2\partial_{\xi\eta} + \partial_{\eta\eta}\right)u$

$$- c^2\left(\partial_{\xi\xi} + 2\partial_{\xi\eta} + \partial_{\eta\eta}\right)u = 0$$

$$\Rightarrow \quad -4c^2\frac{\partial^2 u}{\partial\xi\partial\eta} = 0 \quad \Rightarrow \text{ need } \frac{\partial^2 u}{\partial\xi\partial\eta} = 0.$$

Integrating in ξ $\Rightarrow$ $\frac{\partial u}{\partial\eta} = r(\eta)$

$$\Rightarrow \quad u = p(\xi) + \underbrace{\int^{\eta} r(s)\cdot ds}_{q(\eta)}$$

$\boxtimes$

How to solve I.V.P ? $\begin{cases} u(0,x) = p(x) + q(x) = f(x) & (1) \\ u_t(0,x) = -cp'(x) + cq'(x) = g(x) & (2) \end{cases}$

$(1) \Rightarrow p'(x) + q'(x) = f'(x)$

Combine with (2) $\Rightarrow 2cp'(x) = cf'(x) - g(x)$

$$\Rightarrow \quad p(x) = \frac{1}{2}f(x) - \frac{1}{2c}\cdot\int_0^x g(s)\cdot ds + a$$

integ. constant

$(1) \Rightarrow q(x) = f(x) - p(x)$

$$= \frac{1}{2}f(x) + \frac{1}{2c}\int_0^x g(s)\cdot ds - a$$

The final solution

$$u(t,x) = p(\xi) + q(\eta) = \frac{1}{2}\left[f(\xi) + f(\eta)\right] - \frac{1}{2c}\int_0^{\xi} g(s)\cdot ds + \frac{1}{2c}\int_0^{\eta} g(s)\cdot ds$$

$$u(t,x) = \frac{f(\xi)+f(\eta)}{2} + \frac{1}{2c}\int_{\xi}^{\eta} g(s)\cdot ds$$

THEOREM: The solution of the IVP $\begin{cases} u_{tt} = c^2 u_{xx}. \\ u(0,x) = f(x) \\ u_t(0,x) = g(x) \end{cases}$

is given by,

$$u(t,x) = \frac{f(x-ct) + f(x+ct)}{2} + \frac{1}{2c}\int_{x-ct}^{x+ct} g(s)\cdot ds,$$

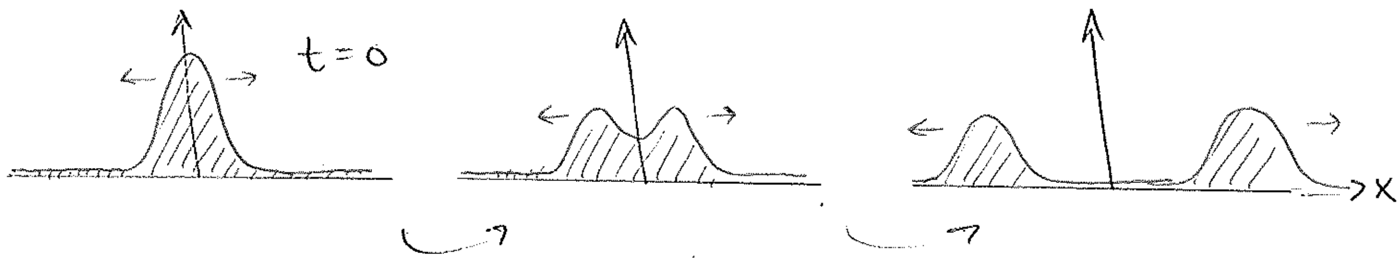
known as d'Alembert's solution to the wave equation.

Example: Suppose $g(x) \equiv 0$ (no initial velocity). Then the motion of the string is entirely due to initial displacement,

$$u(t,x) = \underbrace{\frac{1}{2} f(x-ct) + \frac{1}{2} f(x+ct)}$$

initial displacement splits into two waves with half the height.

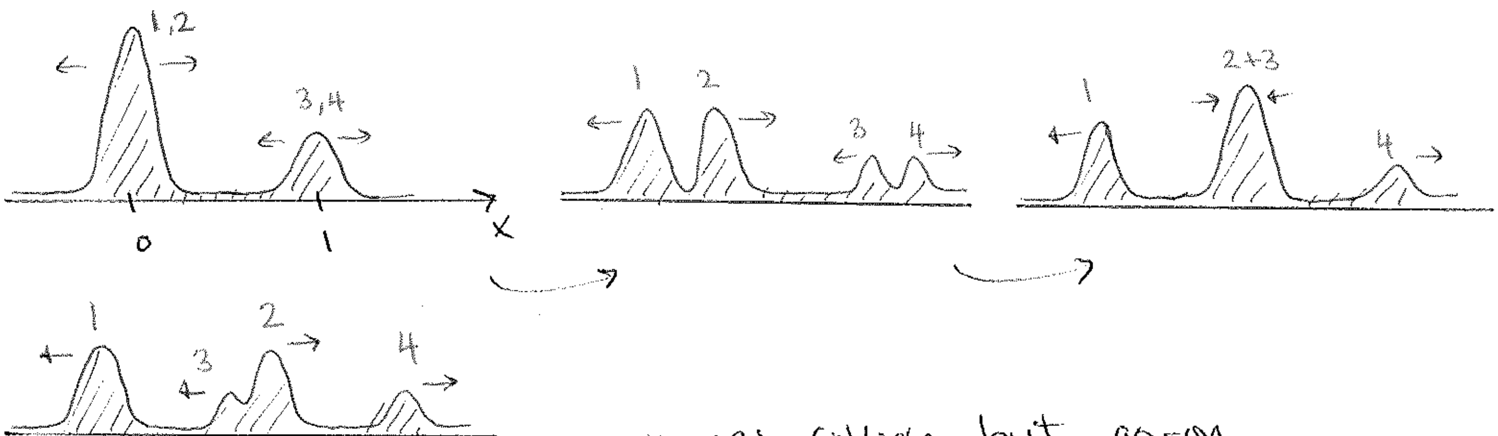
If $f(x) = e^{-x^2}$



t increasing

$$u(t,x) = \frac{1}{2} e^{-(x-ct)^2} + \frac{1}{2} e^{-(x+ct)^2}$$

If $f(x) = 2e^{\frac{-x^2}{\epsilon}} + e^{-\frac{(x-1)^2}{\epsilon}}$, $\epsilon = 0.1$



→ waves collide, but go-on, unchanged after interaction

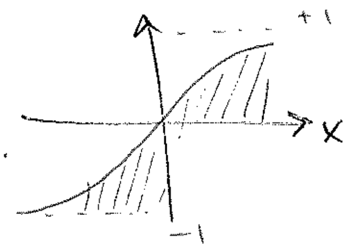
→ notice that if initial $f(x)$ is localised, i.e. $f(x) \to 0$ as $x \to \pm\infty$, then it is easy to see the waves. But not always easy to spot the distinct left/right waves!

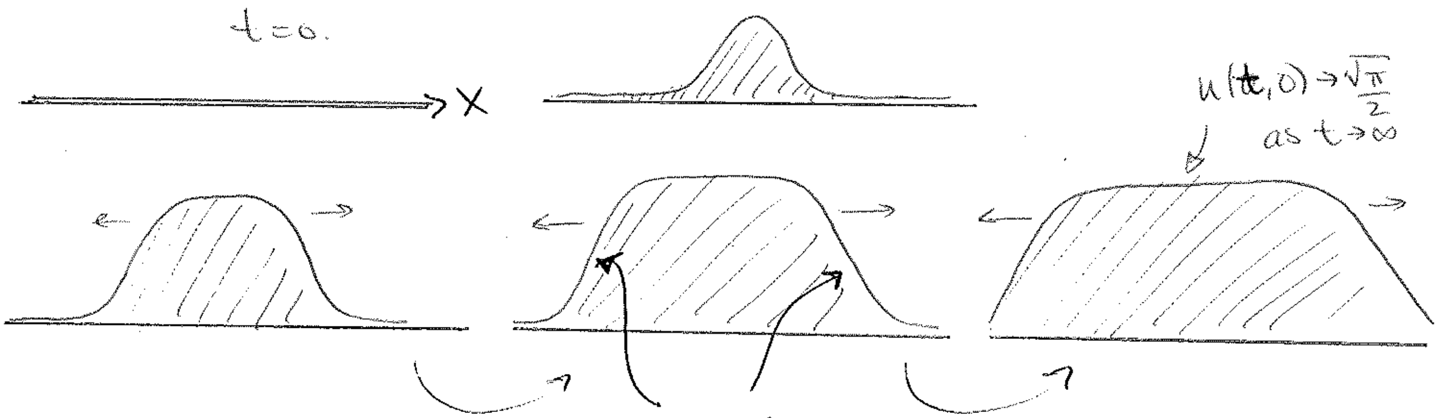
Example: Assume $f(x) \equiv 0$ and the motion is entirely due to $u_t(0,x) = g(x)$ (e.g. hitting a string).

If $u_t(0,x) = g(x) = e^{-x^2}$,

$$u(t,x) = \frac{1}{2c}\int_{x-ct}^{x+ct} e^{-s^2}\cdot ds = \frac{\sqrt{\pi}}{4c}\left\{\operatorname{erf}(x+ct) - \operatorname{erf}(x-ct)\right\}$$

$$\operatorname{erf}(x) = \text{error function} = \frac{2}{\sqrt{\pi}}\int_0^x e^{-s^2}\cdot ds.$$





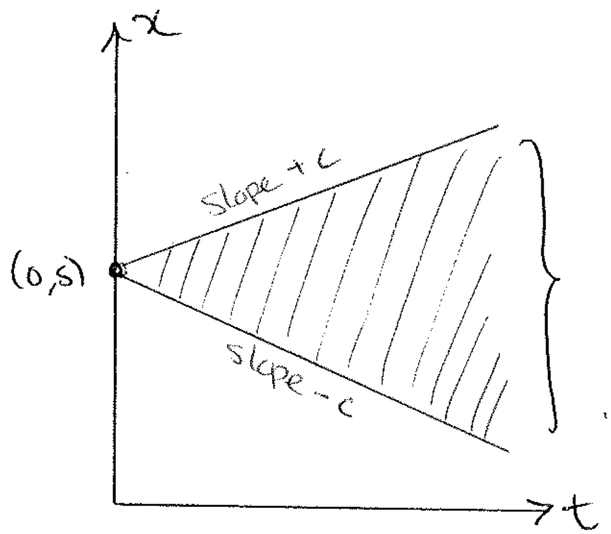
So initial velocity leaves the string permanently deformed.

The lines of slope $\pm c$, where the characteristics variables are constant,

$$\begin{cases} \xi = x - ct \\ \eta = x + ct \end{cases}$$

are the characteristics of the wave equation.

The second order wave equation thus has two distinct characteristics passing through (t,x).



We call this wedge the domain of influence of the point $(0,s)$.

Change the I.C. @ $(0,s)$ will affect solutions along the characteristic (if you change $u(0,s)$), and will affect solutions within the wedge (if you change $u_t(0,x)$). Information is contained (or propagated) in domain of influence.