LECTURE 16.

The nonlinear PDE:

$$\boxed{u_t + u u_x = 0}$$

provides a simple example of nonlinear transport where

→ wave speed depends on elevation

→ waves of elevation move right
" " depression " left

→ used to model e.g. overturning waves.

Can we use the method of characteristics?

$$\Rightarrow \text{ need } \quad \frac{dx}{dt} = u(t,x) \qquad (*)$$

so to find $x = x(t)$, we need $u(t,x)$
but to find u, need $x(t)$!

But note that along characteristics $u(t, x(t)) = h(t)$,

$$\frac{dh}{dt} = \frac{du(t,x(t))}{dt} = \left(\frac{\partial u}{\partial t} + u \cdot \frac{\partial u}{\partial x}\right)_{x = x(t)} = 0.$$

∴ PDE has a solution which is still constant along characteristics. Characteristics have $\frac{dx}{dt} = \text{const.}$ (are straight lines). To solve IVP:

$$u(0,x) = f(x),$$

We note that the full solution is,

$$u = f(\underbrace{x - tu})$$

since the solution is be a function of characteristic variable $\xi = x - tu$.

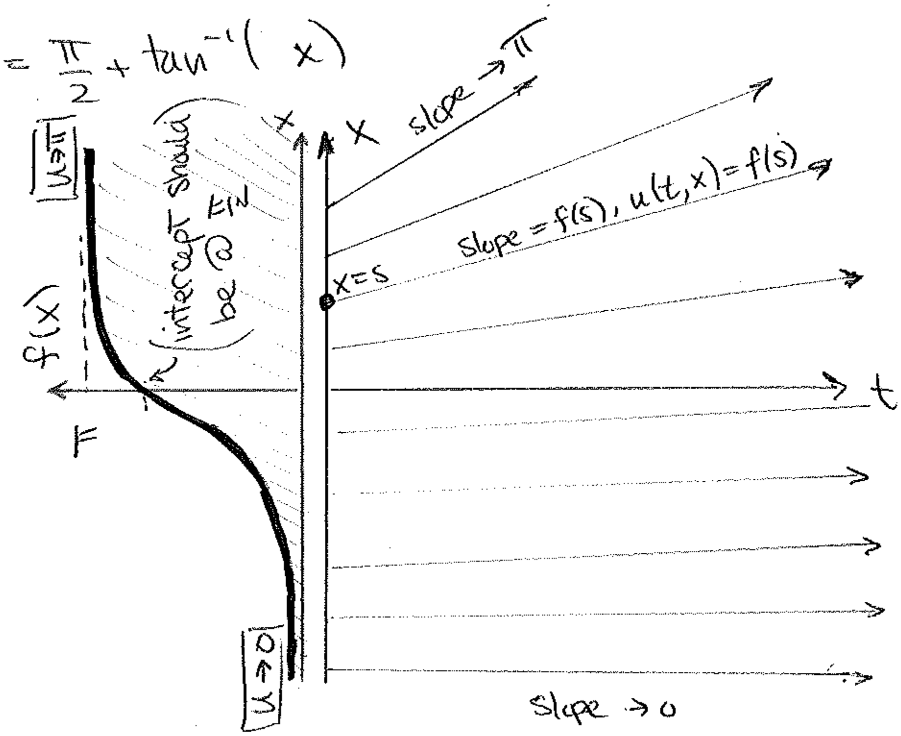
Through each point $(0, s)$ on the x-axis, draw the characteristic line $x = t \cdot f(s) + s$.

Along these lines, the solution is $u(t, tf(s)+s) = f(s)$.

(a) What do solutions look like, given a $f(x)$?

(b) Are solutions well-defined?

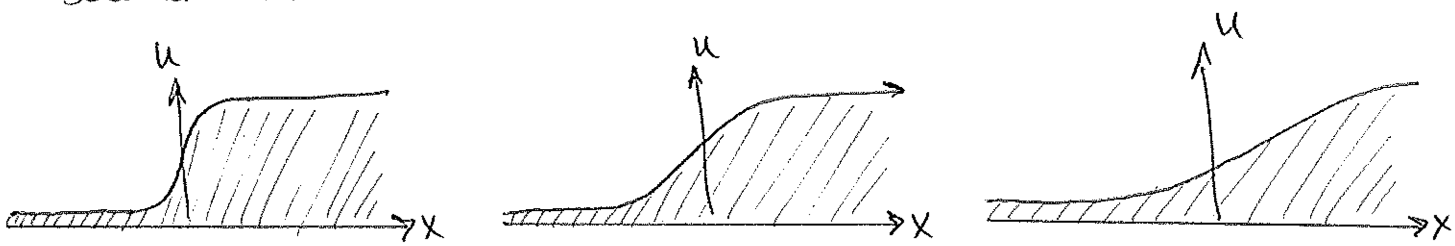
Consider: $u(0,x) = \frac{\pi}{2} + \tan^{-1}(x)$



→ Characteristics fan-out

→ characteristics never cross so solutions begin @ $u(0,x) = f(x)$ and are well-defined $\forall\, t$.

Solutions then look like this:

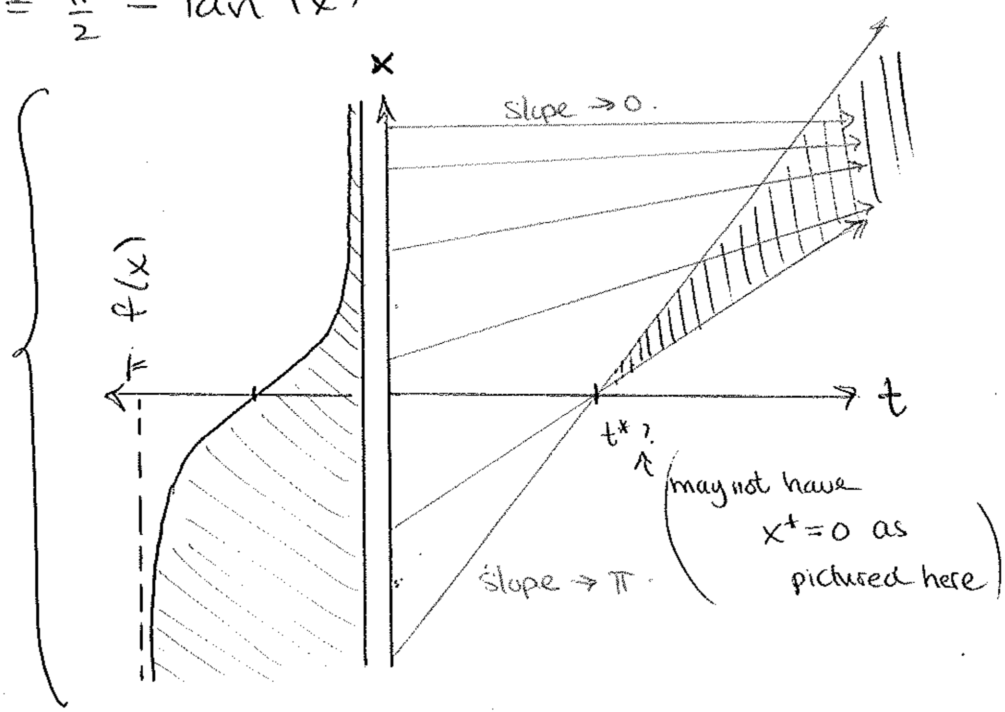


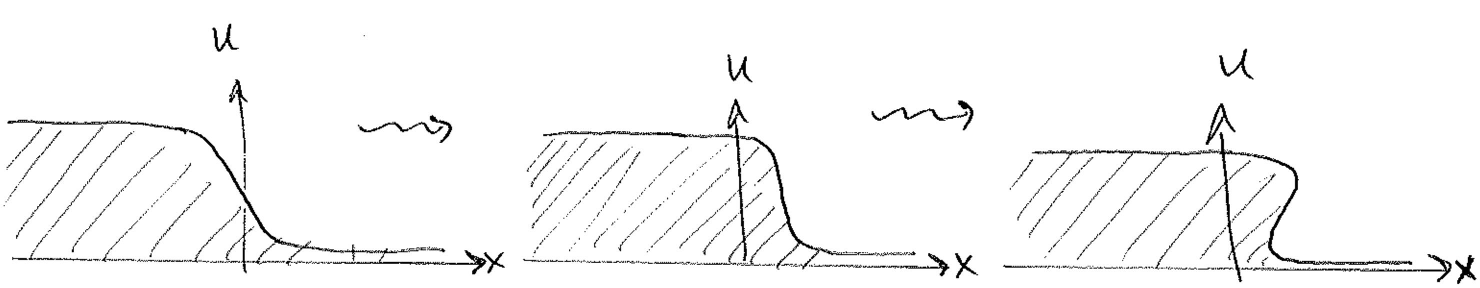
Physically, such solutions represent RAREFACTION (or EXPANSION) WAVES, which spread out as t increases.

Notice that $f'(x) \geq 0 \ \forall x$. What if the I.C. has $f'(x) < 0$? Consider:

$$u(0,x) = \frac{\pi}{2} - \tan^{-1}(x)$$

→ characteristics now cross. At a point where they cross, u takes at least two different values @ the same value of x.





The solution is now called a COMPRESSION WAVE.

Where do solutions first break down? Need to find smallest t^* s.t.

$$\frac{\partial u}{\partial x}(t, x^*) \to \infty \quad \text{as } t \to t^*$$

Since $u(t,x) = f(\xi)$, then, since $\xi = x - tu$,

$$\frac{\partial u}{\partial x} = \frac{\partial}{\partial x} f(\xi) = f'(\xi) \cdot \frac{\partial \xi}{\partial x} = f'(\xi)\left(1 - t \cdot \frac{\partial u}{\partial x}\right)$$

$$\Rightarrow \frac{\partial u}{\partial x} = \frac{f'(\xi)}{1 + t f'(\xi)} \to \infty \quad \text{as } t \to \frac{-1}{f'(\xi)}$$

$\therefore$ If the initial data has $f'(x) < 0$ at some x, then the solution along the characteristic $(0,x)$ will fail to be smooth @ $t = \frac{-1}{f'(x)}$. Thus,

$$t^* = \min\left\{ \frac{-1}{f'(x)} \,\middle|\, f'(x) < 0 \right\}$$

In our case $f(x) = \frac{\pi}{2} - \tan^{-1}(x) \Rightarrow f'(x) = \frac{-1}{1+x^2}$

$\Rightarrow t^* = \min\{1 + x^2\} = 1$ (when $x = 0$). Thus, the earliest breakdown occurs @ $t = 1$, along the characteristic through $(0,0) \Rightarrow x^* = \underset{\uparrow\, 0}{\xi} + t^* u = 0 + 1 \cdot f(0) = \frac{\pi}{2}$

We now have a rather severe problem (since @ the point (x^*, t^*), our solution has 2 different values). What can we do?

Possibility 1) · Avoid choosing I.C.s which produce crossing characteristics (i.e. $f(x) \geq 0 \ \forall x$).

... not physically sensible.

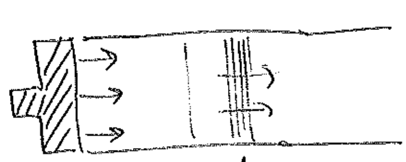
Possibility 2) Assume our solution $u(t,x)$ only works close to $t=0$, but may break down when t is larger.

Possibility 3) Extend the notion of solutions to allow discontinuities $\Longrightarrow$ SHOCK WAVES

SHOCK DYNAMICS.

Shocks occur in many physical applications. Two well-known shocks are for breaking waves, and also in gas dynamics.

Imagine a long pipe with a piston which compresses the gas



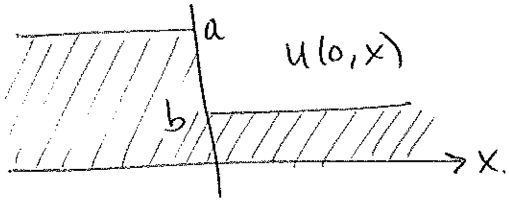
shock.

If the piston moves rapidly enough, gas is compressed and a shock wave forms

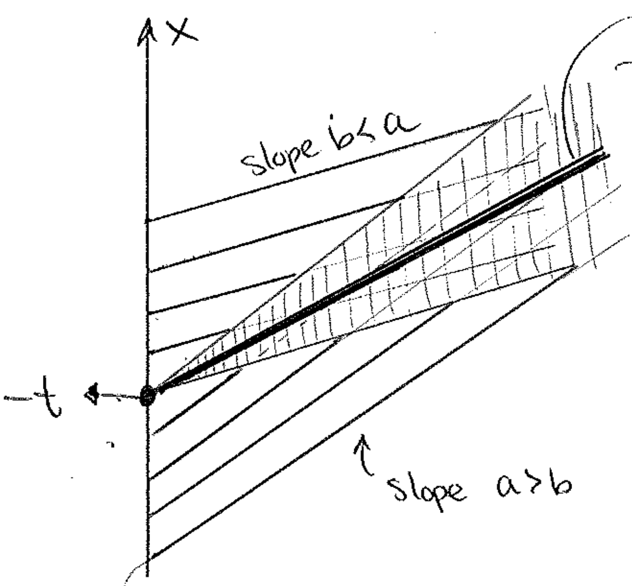
The key issue is whether our mathematical model (the PDE) is even sufficient to describe the necessary physics. After all, we are dealing with functions that are not only <u>not</u> <u>differentiable</u>, but fail to be continuous / single-valued.

But in cases where we have a conservation law (e.g. conservation of cars, mass, energy, etc.) we can work with alternative representations of the law.

<u>Example</u>: Consider the I.V.P.
$$\begin{cases} u_t + u u_x = 0. \\ u(0,x) = \begin{cases} a & x > 0 \\ b & x < 0 \end{cases} \end{cases}$$



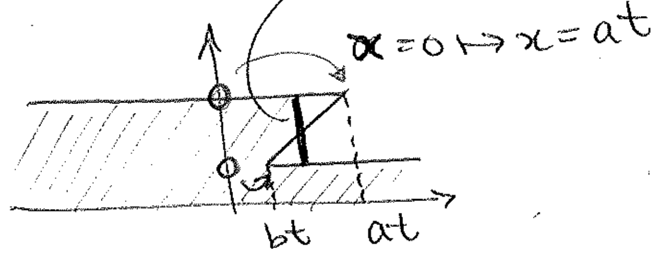
So our initial condition is already discontinuous.



$x = s(t)$

Where do we place the shock in this unknown wedge?

shock solution.



So the question is where we should place the shock. The solution is multivalued within $bt < x < at$, and so we should impose that the solution switches from $u = a$ to $u = b$ at some $x = s(t)$ ... but how to determine $s(t)$?

Let us go back to our traffic flow model.

Recall: $u(t,x)$ = density of cars (cars/m) (previously ρ)
$q(t,x)$ = flux of cars (cars/sec).

and $M(t) = \int_a^b u(t,x) \cdot dx$ = # of cars between $[a,b]$ (mass).

and conservation of mass gave:

$$\underbrace{\frac{d}{dt}\int_a^b u(t,x)\cdot dx}_{\substack{\text{change of cars}\\ \text{in } [a,b]}} + \underbrace{\int_a^b \frac{\partial q(t,x)}{\partial x}\cdot dx}_{\substack{\text{difference in cars}\\ \text{entering } x=a \text{ and exiting}\\ x=b}} = 0.$$

From this, we derived $\frac{\partial u}{\partial t} + \frac{\partial q}{\partial x} = 0$

and different flux-density relations gave different traffic models.

For example: $q(t,x,u) = cu = \text{const} \times u$

gives $\boxed{\dfrac{\partial u}{\partial t} + c \cdot \dfrac{\partial u}{\partial x} = 0}$ Uniform transport.

or $q(t,x,u) = \frac{1}{2}u^2$ gives

$$\frac{\partial u}{\partial t} + \frac{\partial}{\partial x}\left(\frac{1}{2}u^2\right) = \boxed{\frac{\partial u}{\partial t} + u \cdot \frac{\partial u}{\partial x} = 0}$$ simple nonlinear transport

So the question is: instead of working with PDE, can we use the integral form of conservation

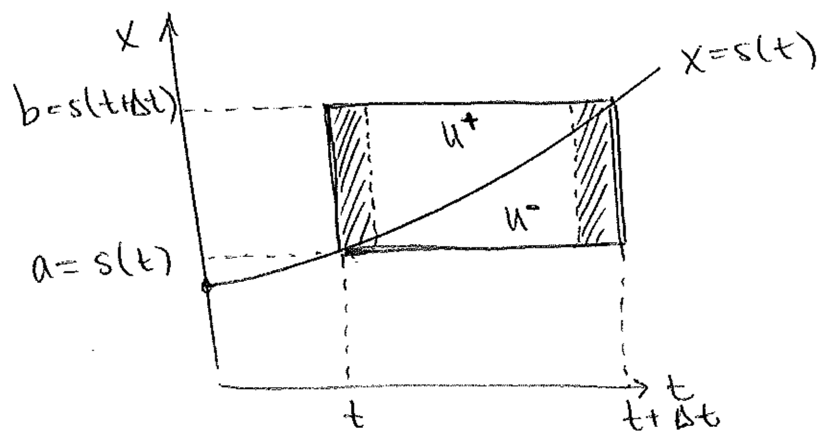
$$\frac{dM}{dt} = -\int_a^b \frac{\partial q}{\partial x} \cdot dx = -\Big\{q(t,b,u(t,b)) - q(t,a,u(t,a))\Big\}$$

$$= -[q]_{x=a}^{x=b} \qquad (*)$$

Where $(*)$ is expressed as a jump, $[f(x)]_{x=a}^{x=b} = f(b) - f(a)$.

Can we use $(*)$ to ensure that wherever the shock is placed, $x = s(t)$, that it conserves mass?

Consider:



We must ensure the mass in $[a,b]$ at time t, when the shock is at $x = s(t)$ is conserved at later time $t+\Delta t$, when the shock is at $x = s(t+\Delta t)$

$$M(t) = \int_a^b u(t,x)\,dx \approx u^+(t)(b-a) = u^+(t)\left\{s(t+\Delta t) - s(t)\right\}$$

$$M(t+\Delta t) = \int_a^b u(t+\Delta t)\cdot dx \approx u^-(t+\Delta t)(b-a) = u^-(t+\Delta t)\left\{s(t+\Delta t) - s(t)\right\}$$

$$\Rightarrow \frac{dM}{dt} = \lim_{\Delta t \to 0} \frac{M(t+\Delta t) - M(t)}{\Delta t} = \lim_{\Delta t \to 0}\left[\left\{u^-(t+\Delta t) - u^+(t)\right\} \times \left\{\frac{s(t+\Delta t) - s(t)}{\Delta t}\right\}\right]$$

since $u^-(t) \neq u^+(t)$ (by the charact. crossing)

$$\frac{dM}{dt} = \left\{u^-(t) - u^+(t)\right\}\frac{ds}{dt} = -[q]_{x=s(t)}^{x=s(t+\Delta t)}$$

$\therefore$ Letting $\Delta t \to 0$, $\Rightarrow$ $\dfrac{ds}{dt} = \dfrac{[q]_-^+}{[u]_-^+}$

where $[\,\cdot\,]_-^+ = [\,\cdot\,]_{x=s^-(t)}^{x=s^+(t)}$

THEOREM: Let $u(t,x)$ be the solution of a nonlinear conservation equation:

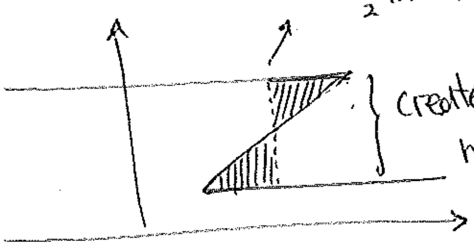
$$\frac{\partial u}{\partial t} + \frac{\partial q}{\partial x} = 0$$

Where $q = q(t,x,u)$ = flux. If u has a discontinuity @ $x = s(t)$, then to maintain conservation of mass across the shock, the $s(t)$ must satisfy the RANKINE-HUGONIOT jump conditions

$$\underbrace{\boxed{\frac{ds}{dt} = \frac{[q]_-^+}{[u]_-^+}}}$$

necessary but not sufficient condition for unique solution.

$x = s(t)$
$= \frac{1}{2}(a+b)\cdot t$



creates equal mass

For the example
$$\begin{cases} u_t + u u_x = 0 \\ u(0,x) = \begin{cases} a, & x < 0 \\ b, & x > 0 \end{cases} \end{cases} \qquad a > b$$

$$\frac{ds}{dt} = \frac{\left[\frac{1}{2}u^2\right]_-^+}{[u]_-^+} = \frac{\frac{1}{2}\left(u^{+2} - u^{-2}\right)}{(u^+ - u^-)} = \frac{1}{2}\left(u^+ + u^-\right)$$

since $u^+ = b$, $u^- = a$, then the resultant shock solution is,

$$u(t,x) = \begin{cases} a & x < ct \\ b & x > ct \end{cases} \qquad c = \left(\frac{a+b}{2}\right)$$

→ travels with an average velocity between two speeds (FIG.)