LECTURE 15

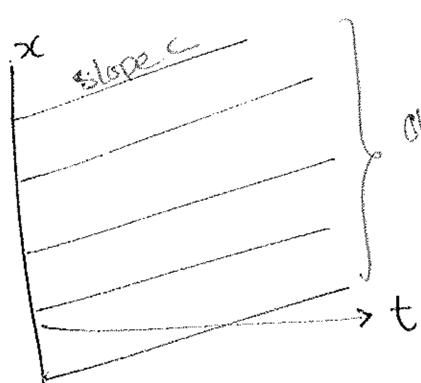
TRANSPORT AND WAVES.

In the last class, we showed how to solve equations of the form: $au_x + bu_y = 0$ by a geometric method using characteristics. In fact, in the previous lectures, we showed how, if $u = u(x)$ is concentration (e.g. of cars), then

$$\boxed{\frac{\partial u}{\partial t} + c \cdot \frac{\partial u}{\partial x} = 0,}$$

is a TRANSPORT EQUATION, describing the transport of a substance moving with uniform velocity, c.

Since PDE $\Rightarrow (1, c) \cdot (u_t, u_x) = 0$, then we expect characteristics along $x = c \cdot t + \text{const}$.



characteristics, where the solution $u(t,x)$ is constant.

(I have graphed tx instead of xt for a good reason)

$$\therefore u(t,x) = u(x - ct)$$

Instead of geometry, we can use algebraic method.

Let $\underset{\substack{\nearrow \\ \text{moving frame of reference.}}}{\xi} = \underset{\substack{\uparrow \\ \text{fixed frame.}}}{x} - ct$

Now, let $u(t,x) = v(t,\xi)$. By chain rule:

$$\frac{\partial u}{\partial t} = \frac{\partial v}{\partial t} + \frac{\partial \xi}{\partial t}\frac{\partial v}{\partial \xi} = \frac{\partial v}{\partial t} - c\frac{\partial v}{\partial \xi}$$

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial \xi}$$

$$\therefore \text{PDE} \Rightarrow \left[\frac{\partial v}{\partial t} - c\frac{\partial v}{\partial \xi}\right] + c\frac{\partial v}{\partial \xi} = \boxed{\frac{\partial v}{\partial t} = 0}$$

$$\therefore v = v(\xi)$$

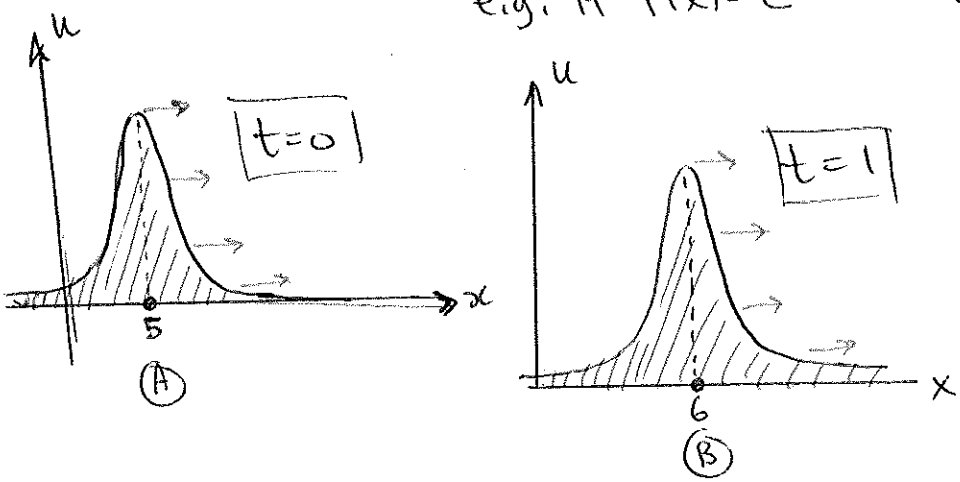
only a function of ξ

$$\therefore u(t,x) = v(\xi) = v(x-ct)$$

(Somewhat abuse in notation)

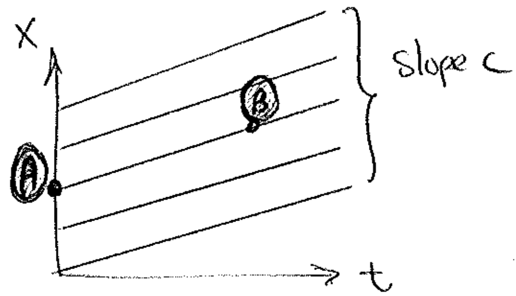
Notes: → If we provide an initial profile $u(0,x) = f(x)$, then the solution is $f(x-ct)$

e.g. if $f(x) = e^{-(x-5)^2}$ and $c = 1$



so the profile steadily translates rightwards @ speed c.

→ The height of the profile, u, at any given point and time (t,x), only depends on which characteristic passes through the point



We say that:

$\boxed{\text{SIGNALS PROPAGATE ALONG CHARACTERISTICS}}$

→ note that any disturbance in the initial data $u(0,x)$ only affects points along its characteristic.

Transport with decay:

The PDE: $\boxed{\dfrac{\partial u}{\partial t} + c\dfrac{\partial u}{\partial x} + au = 0}$

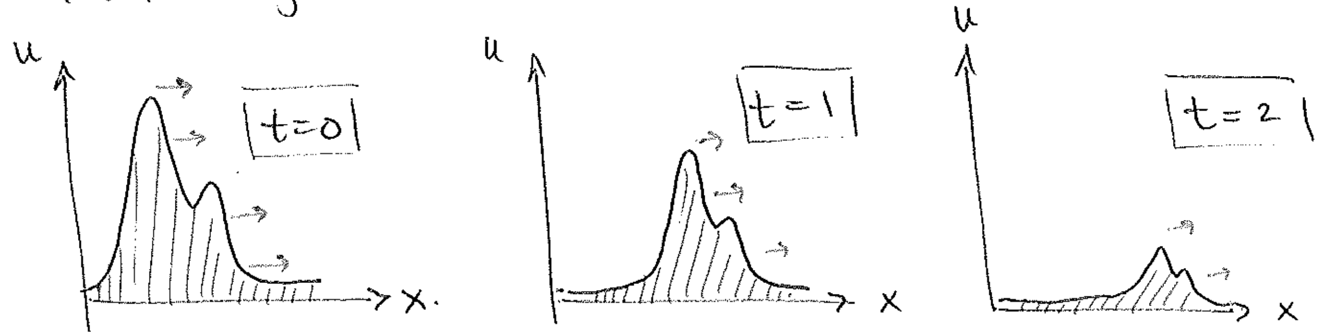
where $a, c > 0$ models transport of a decaying substance in a uniform flow with speed c and rate of decay a.

If we again write $u(t,x) = v(t,\xi) = v(t, x-ct)$, we get

$$\frac{\partial v}{\partial t} + av = 0 \Rightarrow \frac{d}{dt}\left(ve^{at}\right) = 0$$

$$\Rightarrow v = f(\xi)e^{-at}$$

Thus, the general solution is $u(t,x) = f(x-ct)\, e^{-at}$



→ notice that solution is no longer constant on characteristics spanned by lines $x - ct = \xi$, but if we write:

$$(1, c) \cdot \nabla u = -au$$

s = local coordinate along characteristics $\quad \dfrac{du}{ds} = -au \Big\}$ you can see where the exp. comes from

We will return to this later!

Non-uniform transport

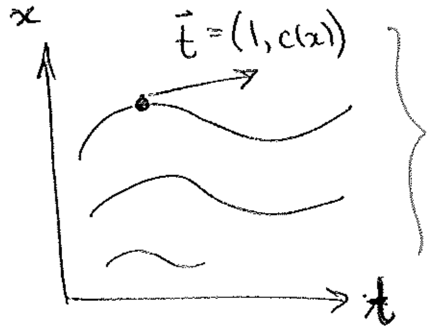
Consider now $\quad \dfrac{\partial u}{\partial t} + c(x) \cdot \dfrac{\partial u}{\partial x} = 0$

speed of transport now dependent on position (e.g. dense roads vs. open roads)

Method 1: (Geometric)

Since $\quad PDE = (1, c(x)) \cdot (u_t, u_x) = 0$

Solutions are constant along characteristic curves whose tangents are $\vec{t} = (1, c(x))$



If such curves are given by $x = x(t)$, then we need $\boxed{\dfrac{dx}{dt} = c(x)}$

So now we need to solve this differential equation

Method 2 : (Algebraic)

Let us search for the curves $x = x(t)$ such that along these curves, the solution,

$$u(t, x(t)) = h(t)$$

is constant. Then

$$\frac{dh}{dt} = \frac{d}{dt} u(t, x(t)) = \frac{\partial u}{\partial t}(t, x(t)) + \frac{\partial u}{\partial x}(t, x(t)) \cdot \frac{dx}{dt}$$

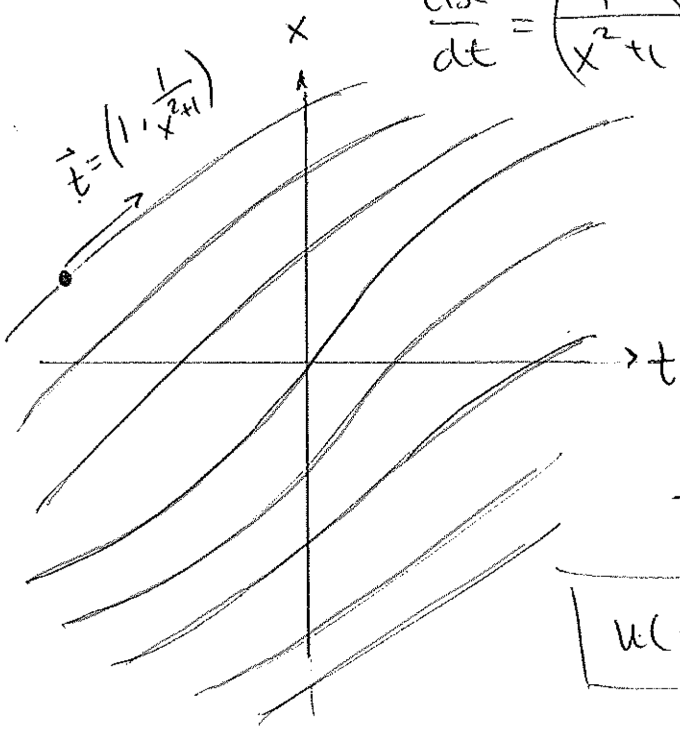
note 'd' instead of '∂'

Choose $\dfrac{dx}{dt} = c(x)$ so that the PDE $= 0$.

$\therefore \dfrac{dh}{dt} = 0$ along the charac. curves $\dfrac{dx}{dt} = c(x)$

Example: Solve $\dfrac{\partial u}{\partial t} + \left(\dfrac{1}{x^2+1}\right)\cdot\dfrac{\partial u}{\partial x} = 0.$

The eqn. for the characteristics is



$$\frac{dx}{dt} = \left(\frac{1}{x^2+1}\right) \Rightarrow (x^2+1)\cdot dx = dt$$

$$\Rightarrow \frac{1}{3}x^3 + x + \text{const} = t$$

$$\Rightarrow \frac{1}{3}x^3 + x = t + k$$

const

Thus, the solution is

$$\boxed{u(t,x) = f\left(\tfrac{1}{3}x^3 + x - t\right)}$$

What to do if we are given initial data $u(0,x) = \dfrac{1}{1+[x+2.75]^2}$?

need to solve $f\left(\frac{1}{3}x^3+x\right) = \dfrac{1}{1+(x+2.75)^2}$ ... not very helpful!

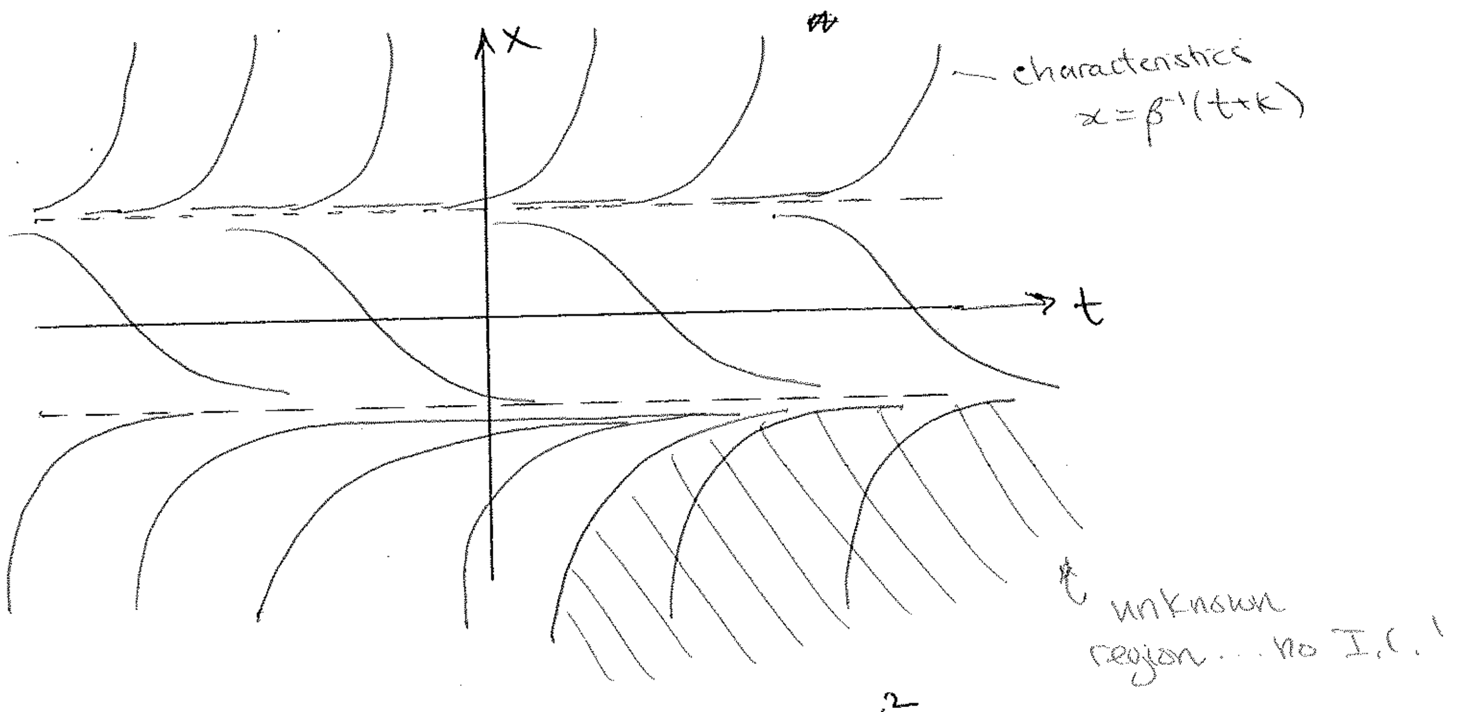
But with this implicit form of the solution, the solution can be computed numerically. (See Matlab). What we see is a wave that moves to the right (since $c(x) > 0 \ \forall x$) and gets progressively narrower as $c(x) \to 0$. (slower)

Example: Consider $u_t + (x^2-1)u_x = 0$. (non unif. transport)

Characteristics $\Rightarrow$ $\dfrac{dx}{dt} = (x^2-1)$

$$\Rightarrow \underbrace{\int \frac{dx}{(x-1)(x+1)}}_{=\frac{1}{2}\log\left|\frac{x-1}{x+1}\right|} = t+K.$$

Let us write $\beta(x) = \frac{1}{2}\log\left|\frac{x-1}{x+1}\right| = t+K$ so that characteristics are given by $x = \beta^{-1}(t+K)$.



Consider now the I.C. $u(0,x) = e^{-x^2}$

Then the I.C. uniquely prescribes the solution as long as the characteristics intersect the x-axis.

We can in fact solve: $x = \dfrac{1+e^{2(t+K)}}{1-e^{2(t+K)}}$ for $|x|>1$

The critical curve has $x \to -\infty$ as $t \to 0^+ \Rightarrow k=0$.

Then the shaded region is

$$\underbrace{x \geqslant \frac{1+e^{2t}}{1-e^{2t}} \quad \text{for } t>0}$$

solution is not prescribed in this region, an we can arbitrarily set it to be zero.

We can show that for $t \geqslant 0$ the solution is

$$u(t,x) = \exp\left\{-\left(\frac{(x+1)e^{2t}+x-1}{(x+1)e^{2t}-x+1}\right)^2\right\}$$

within the known regions. See Matlab sheet.

As $t \to \infty$, solution converges non-uniformly to step

$$u(t,x) \to \begin{cases} \frac{1}{e} & x \geqslant -1 \\ 0 & x < -1 \end{cases} \quad \text{as } t \to \infty$$

General remarks for transport eqn: $u_t + c(x)u_x = f(x,y,u)$

→ Characteristics are given by $\frac{dx}{dt} = c(x)$

By exist + unique of ODES, $\exists$ a unique solution passing through (t,x) [assuming $c(x)$ is cts. and Lipschitz].

→ Characteristic curves cannot cross

→ If $t = \beta(x)$ is a characteristic curve, then so is its horizontal translate: $t = \beta(x) + k \ \forall k$

→ Waves cannot reverse in direction since each characteristic is a strictly monotone function

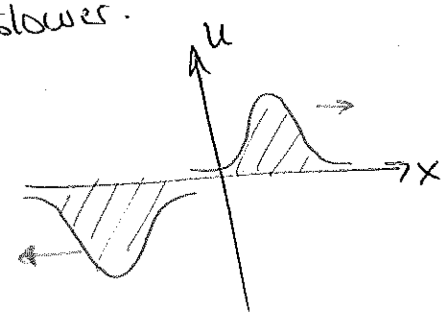
→ As $t \to \infty$, characteristics either asymptote to a fixed point $(c(x^*) = 0)$ or has $x \pm \infty$.

NONLINEAR TRANSPORT.

The canonical nonlinear PDE that describes nonlinear transport is:

$$\boxed{u_t + u u_x = 0}$$

→ Wave speed now depends on height $(c = u)$ ⇒ larger waves move faster and smaller waves move slower.



→ Waves of elevation move right
" " depression " left

→ used to model: traffic flow, shock waves in gas dynamics, floods and water waves.

Can we use the same method of characteristics?

The problem is that we now need

$$\frac{dx}{dt} = u(t,x) \qquad (*)$$

for the equation $x = x(t)$ of the characteristics.

So to find characteristics, we need u, but to find u, we need characteristics!

However, note that along $x = x(t)$, where $u(t,x)\big|_{x=x(t)} = h(t)$,

then,

$$\frac{d}{dt}h(t) = \frac{d}{dt}u(t,x(t)) = \frac{\partial u}{\partial t}\bigg|_{x=x(t)} + \frac{dx}{dt}\cdot\frac{\partial u}{\partial x}\bigg|_{x=x(t)}$$

$$= \left(\frac{\partial u}{\partial t} + u\cdot\frac{\partial u}{\partial x}\right)_{x=x(t)} = 0 \text{ by PDE}$$

$\therefore \frac{dh}{dt} = 0$ along characteristics, so the solution is still constant along characteristics. Thus $(*)$ is const. when $x = x(t)$.

$$\Rightarrow \underline{x(t) = u(t,x(t))\cdot t + k.}$$

Characteristic curves are straight lines whose slope equals the value of u on the line

$\Rightarrow$ The solution must be $\boxed{u(t,x) = f(\xi) = f(x-ut)}$