§ DOMAIN OF DEFINITION AND INITIAL/BOUNDARY CONDITIONS

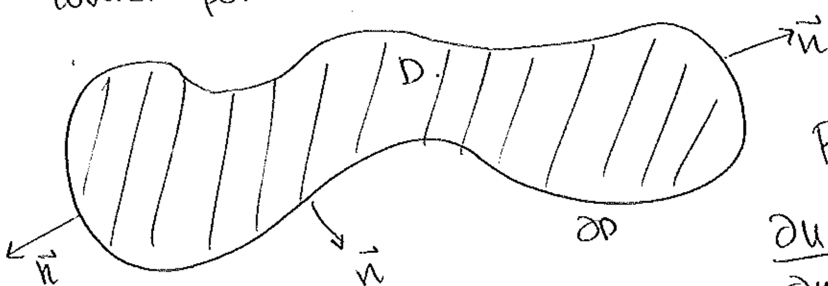
Just as for the case ODEs, a PDE needs either initial or boundary conditions to specify an exact, particular solution. These are generally argued on physical grounds, but we must be concerned with the fact that some PDEs may require boundary conditions (and not initial conditions) or vice-versa.

Also, we should take care to define the appropriate domain of definition, D, of the PDE.

The three most important B.C.s are:

[D] Dirichlet condition : Specify u on ∂D.
[N] Neumann condition: Specify $\frac{\partial u}{\partial n}$ on ∂D.
[R] Robin condition: Specify $\frac{\partial u}{\partial n} + au$ on ∂D.

where $\vec{n} = (n_1, n_2, n_3)$ is the unit normal along ∂D. which points outwards



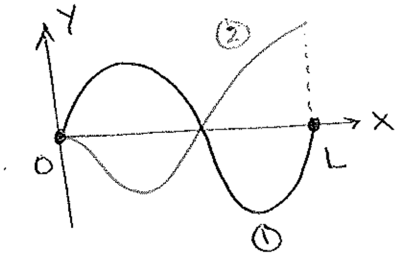
Recall that to compute

$$\frac{\partial u}{\partial n} = \vec{n} \cdot \nabla u = (n_1, n_2) \cdot \left(\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}\right)$$

(for 2D)

Examples:

Vibrating string: $D = [0, L]$ and $\boxed{u_{tt} = c^2 u_{xx}}$



① Fixed ends, (like a violin string)

$$[D]: y(0,t) = 0 = y(L,t)$$

② Free end

Since the tension is approx. $\frac{dy}{dx}$, then zero tension @ the end $\Rightarrow$

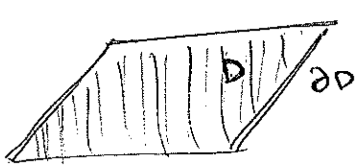
$$[D] \quad y(0,t) = 0$$
$$[N] \quad y_x(L,t) = 0.$$

Note that we will also need to specify an initial profile : $y(x,0) = y_0(x)$.

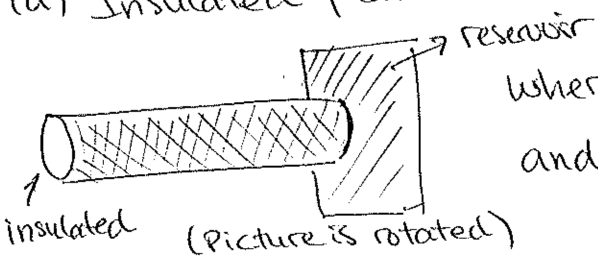
Heat conduction : Heat conduction is described (see your problem set or lecture notes) by the PDE:

$$\boxed{u_t = \alpha \nabla^2 u}$$

where $u = u(x,y,t)$ = temperature and $\alpha > 0$ is the thermal diffusivity.



(a) Insulated plate.



(Picture is rotated)

(b) Long rod dipped in a reservoir

(a) Insulated plate (i.e. a plate heated up to some temperature, and then the ends are insulated)

→ Can show (by Fourier's Law, see your PS) that

insulation $\Rightarrow$ $\frac{\partial u}{\partial n} = 0$ on $(x,y) \in \partial D$.

i.e. Neumann B.C.s.
with $D = \{(x,y) : 0 \le x \le L, 0 \le y \le L\}$.

(b) Consider a long thin rod. Now need to solve:

$$u_{tt} = \alpha u_{xx}$$

on $D = \{x : 0 \le x \le L\}$.

If one end is insulated and the other dipped into a bath, then

$$\frac{\partial u}{\partial x} = \frac{\partial u(0,t)}{\partial n} = 0 \quad \text{[insulated} \Rightarrow \text{Neumann]}.$$

and also:

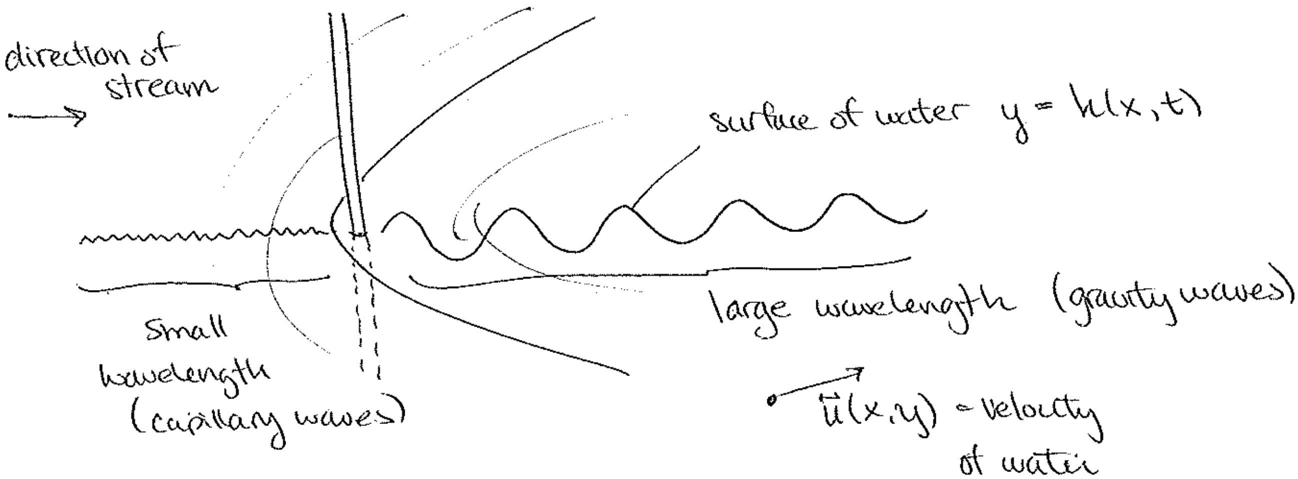
[R] = Robin B.C. = $\frac{\partial u}{\partial x}(L,t) = -\beta.\{u(L,t) - g(t)\}$

temperature of reservoir.

Recall's Newton's law of cooling: The rate of change of the temperature of an object is $\propto$ to the difference between it's own temperature and the ambient temperature.

$$\begin{cases} u_{tt} = \alpha u_{xx} \\ u_x = 0 \ @ \ x = 0 \\ u_x = \beta\{u - g\} \ @ \ x = L \\ u(x,0) = u_0(x) \ @ \ t = 0. \end{cases}$$

Other types of boundary conditions are possible. For example, one scenario is the imposition of conditions at ∞. Consider water flow past a fishing line.



If you observe this in nature, you find two kinds of waves,

$$y_{cap} = A\sin(k_1 x - c_1 t)$$
$$y_{grav} = B\sin(k_2 x - c_2 t)$$

What we observe is y_{cap} ahead of the object, but y_{grav} behind. But how do we even impose this <u>behavioural</u> condition ("unsolved" problem).

Finally, we may have to impose <u>jump conditions</u>. Consider the heat equation over a domain $D = D_1 \cup D_2$ which describes 2 different materials

WELL-POSED PROBLEMS

Given a PDE with a set of initial and boundary conditions we say that the problem is <u>well-posed</u> if it satisfies the following conditions:

(i) <u>Existence</u>: There exists at least one solution, $u(x,t)$ satisfying all these conditions.

(ii) <u>Uniqueness</u>: There is at most one solution

(iii) <u>Stability</u>: The unique solution $u(x,t)$ depends in a stable manner on the data of the problem. (i.e. if the data are changed a little, the corresponding solution changes only a little).

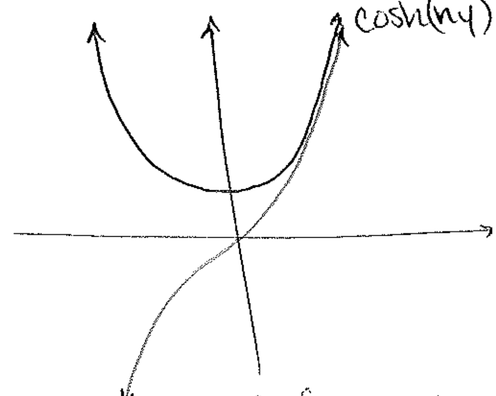
Why is (iii) important? Imagine trying to solve a PDE with some initial condition $u(x,0) = u_0(x) + \epsilon$ where $\epsilon \ll 1$. If the PDE is unstable with respect to this I.C., then no matter how small ϵ is, you would not recover the 'true' solution. (Recall the problem with your problem set and the numerical sol'n of the ODE $y' = 5y - 6e^{-x}$).

<u>Example</u>: Consider Laplace's Eqn: $\nabla^2 u = u_{xx} + u_{yy} = 0$ in the region $D = \{(x,y): -\infty < x < \infty, 0 \le y < \infty\}$. where we specify <u>both</u> $u(x,0)$ and $u_y(x,0)$

We verify that $u_n(x,y) = \frac{e^{-\sqrt{n}}}{n} \cdot \sin(nx)\sinh(ny)$ is a solution of $\nabla^2 u = 0$.

<u>NB</u>: $\sinh(ny) = \frac{e^{ny} - e^{-ny}}{2}$ $\quad \cosh(ny) = \frac{e^{ny} + e^{-ny}}{2}$

and $\frac{d}{dy}\sinh(ny) = \cosh(ny)$ $\quad \frac{d}{dy}\cosh(ny) = \sinh(ny)$



$$\Rightarrow \begin{cases} \frac{\partial^2 u_n}{\partial x^2} = -n e^{-\sqrt{n}} \sin(nx)\sinh(ny) \\ \frac{\partial^2 u_n}{\partial y^2} = n e^{-\sqrt{n}} \sin(nx)\sinh(ny) \end{cases}$$

$$\Rightarrow \nabla^2 u_n = 0.$$

Notice that $u_n(x,0) = 0$.

$$\frac{\partial}{\partial y} u_n(x,0) = e^{-\sqrt{n}} \sin(nx) \to 0 \text{ as } n \to \infty.$$

So let us choose a n very large so that $\frac{\partial u_n}{\partial y}(x,0) = \epsilon$

very small.

Consider solving the problem $\begin{cases} \nabla^2 u = 0. \\ u(x,0) = 0 = u_y(x,0) \end{cases}$

$y \equiv 0$ is a solution.

But since $u_n(x,y) \not\equiv 0$ for $y \neq 0$ and $n \to \infty$, then we have discovered that (one of) the solution(s) of

$$\begin{cases} \nabla^2 u = 0 \\ u(x,0) = 0 \\ u_y(x,0) = \epsilon \end{cases}$$

is nowhere "near" the solution with $\epsilon = 0$.

Check: $|u_n(x,y)| = \frac{e^{-\sqrt{n}}}{n} \cdot |\sin(nx)| \cdot |\sinh(ny)|$

but $\sinh(ny) \sim \frac{e^{ny}}{2}$ as $n \to \infty$ and $y \neq 0$.

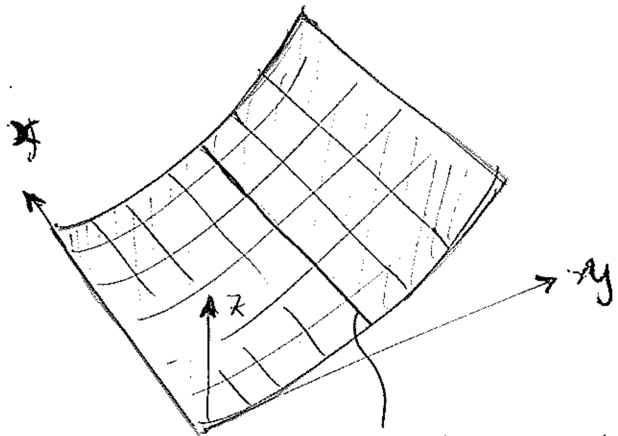
$\therefore |u_n(x,y)| \sim \frac{|\sin(nx)|}{n} e^{-\sqrt{n} + ny} \to \infty$ (as long as $\sin(nx) \neq 0$).

this blows up

§ FIRST ORDER LINEAR EQUATIONS.

Last time, we solved some "baby PDEs": PDEs like $u_x = 0$ ($\Rightarrow u = f(y)$) and for such PDEs, the solution (surface) is constant along lines $y = \text{const.}$ in the xy-plane.

e.g. $u(x,y) = y^2$.



Solution const. along $y = \text{const.}$

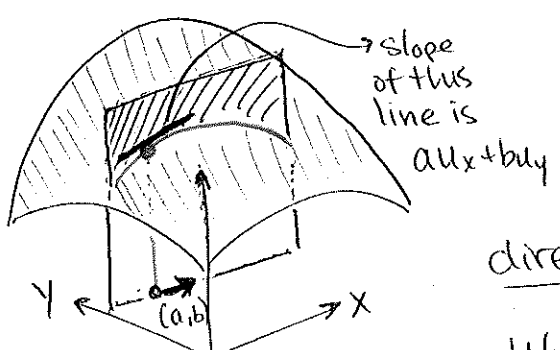
The next simplest class of PDEs is the constant coefficient equation:

$$\boxed{au_x + bu_y = 0}$$ } CONSTANT COEF. FIRST ORDER LINEAR PDE.

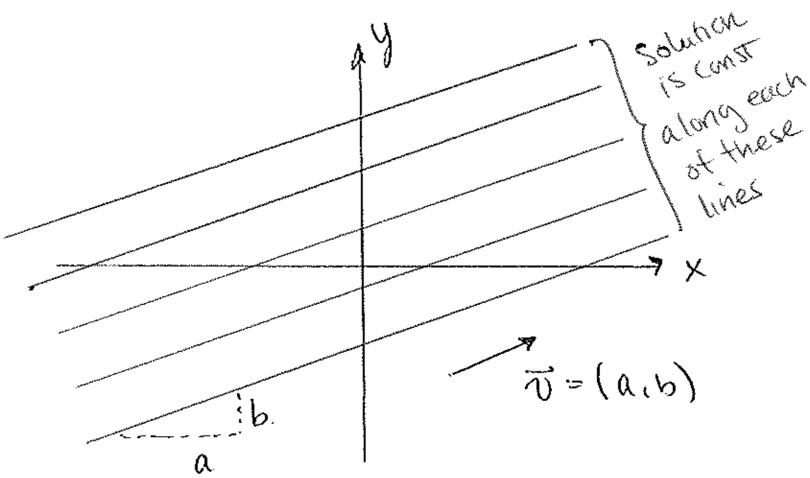
where $a, b \in \mathbb{R}$ not both zero.

Geometric Method:

Since $$au_x + bu_y = (a,b)\cdot\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right)u$$
$$= \underbrace{(a,b)\cdot\nabla u}$$



By vector calculus, this is the directional derivative of the surface $u(x,y)$ along the vector (a,b)



If $\vec{v} = (a,b)$, then

$$\vec{v} \cdot \nabla u = 0$$

and we conclude that along the lines with

$$y = \frac{b}{a}x + \text{const}$$

OR $\boxed{bx - ay = \text{const}}$,

the solution remains constant. These lines parallel to $\vec{v}$ are called characteristics or characteristic lines of the PDE. We must then have the fact that the solution is only a function of

$$bx - ay = \text{const}$$

$$\therefore \; u(x,y) = f(bx - ay) = f(c)$$

where c is arbitrary. The function f will be found via an initial/boundary condition.

Example: Solve the PDE

$$\begin{cases} 4u_x - 3u_y = 0. \\ u(0,y) = y^3 \end{cases}$$

Here $a = 4$, $b = -3$ $\Rightarrow$ $\underbrace{u(x,y) = f(-3x - 4y)}_{\text{general solution of PDE.}}$

$$u(0,y) = f(-4y) = y^3 \Rightarrow \text{if } w = -4y \Rightarrow f(w) = -\frac{w^3}{4^3} = \frac{-w^3}{64}$$

$\therefore\ u(x,y) = \left(\frac{3x+4y}{64}\right)^3.$

(See handout for a plot of the surface and its characteristics).

Now we know how to solve first-order const. coef. linear homogeneous PDEs like $au_x + bu_y = 0$, what about variable coefficient equations like

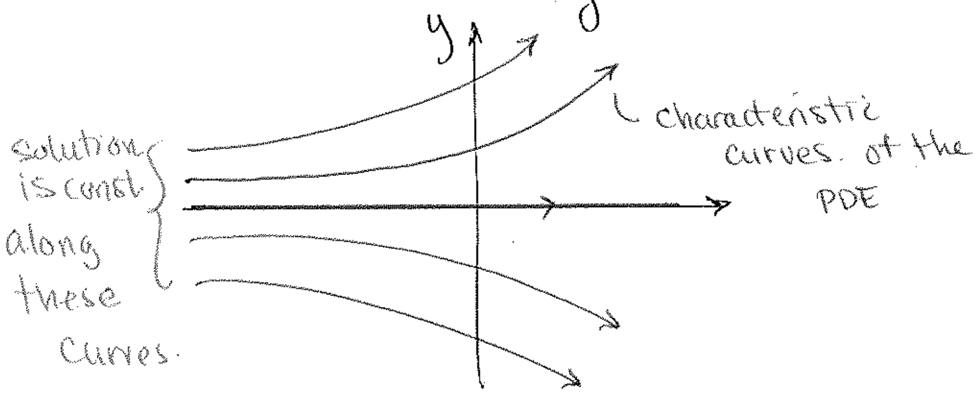
$$\boxed{u_x + y u_y = 0}$$

By the same geometric interpretation $\Rightarrow (1, y) \cdot \nabla u = 0$.

$\therefore$ Along curves whose tangents are given by $\vec{v} = (1, y(x))$, the solution is constant.

$\Rightarrow$ What do these curves look like? Need $\frac{dy}{dx} = \frac{y}{1}$

$$\Rightarrow y(x) = Ce^x$$



Can we check that u is constant along the curves $y = Ce^{x}$? Let the curve(s) be defined by

$$\gamma(s) := (s, Ce^{s})$$

$$\frac{du}{ds} = \frac{dx}{ds}\cdot\frac{\partial u}{\partial x} + \frac{dy}{ds}\cdot\frac{\partial u}{\partial y} = 1\cdot\frac{\partial u}{\partial x} + Ce^{s}\cdot\frac{\partial u}{\partial y} = u_x + y u_y = 0.$$

rate of change of $u(x,y)$ along $(x,y) = (s, Ce^{s})$

We conclude that the general solution is

$$u(x,y) = f(e^{-x}\cdot y) = f(c)$$

for some constant c and f det. by I.C.s/B.C.s.

<u>Example</u>: Solve
$$\begin{cases} u_x + y u_y = 0. \\ u(0,y) = y^3 \end{cases}$$

$$u(x,y) = f(e^{-x}y) \Rightarrow u(0,y) = f(y) = y^3.$$

$$\therefore u(x,y) = (e^{-x}y)^3 = e^{-3x}y^3$$

(See handout for a plot of the solution and its characteristics)