MAR. 28, 2012

LECTURE 13: INTRO TO PARTIAL DIFF. EQNS.

Here are some examples of famous PDEs:

1) Linear transport (advection) equation in 3D:

$$u_t = a_1 u_x + a_2 u_y + a_3 u_z \quad \Big\} \text{ First order linear homogeneous.}$$

a_i const. and $u = u(x,y,z,t)$

2) Laplace equation in 2D.

$$\underset{\uparrow}{\nabla^2 u} = u_{xx} + u_{yy} = 0. \quad \Big\} \text{ Second-order linear homogeneous.}$$

"Laplacian", sometimes written Δu

3) Poisson equation in 3D.

$$\nabla^2 u = u_{xx} + u_{yy} + u_{zz} = -f(x,y,z) \quad \Big\} \text{ Second-order linear inhomog.}$$

4) Heat equation in 3D

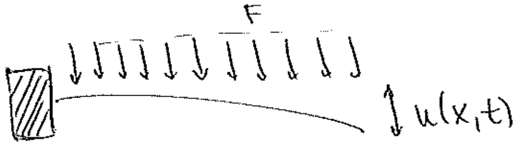
$$u_t = \underset{\substack{\uparrow \\ \text{const.} > 0}}{K} \nabla^2 u + \underset{\substack{\uparrow \\ \text{source term}}}{Q} \quad \Big\} \text{ Second-order linear inhomog.}$$

5) Wave equation in 2D

$$u_{tt} = c^2 \Delta u = \underset{\substack{\uparrow \\ \text{const.}}}{c^2} \nabla^2 u \quad \Big\} \text{ Second-order linear homog.}$$

6) Vibrating beam equation with distributed load in 1D

$$u_{tt} + a^2 u_{xxxx} = F \quad \Big\} \begin{array}{l} \text{Fourth-order} \\ \text{linear.} \\ \text{inhomog.} \end{array}$$



7) Generalized Burger's equation with dissipation

$$u_t + f(u)u_x = \underset{\uparrow \text{ dissipation}}{u_{xx}} \quad \Big\} \begin{array}{l} \text{second-order} \\ \text{quasi-liner.} \quad (*) \\ \cancel{\text{homog}}. \end{array}$$

8) Poisson equation in 3D with nonlinear forcing on u.

$$\nabla^2 u = -f(u) \quad \Big\} \begin{array}{l} \text{second-order} \\ \text{semi-linear} \quad (**) \\ \cancel{\text{homog}}. \end{array}$$

9) Maxwell's equation in a vacuum

$$\begin{cases} \left(\frac{1}{c}\right) \vec{E}_t = \nabla \times \vec{H} \\ \left(\frac{1}{c}\right) \vec{H}_t = -\nabla \times \vec{E} \\ \nabla \vec{H} = \nabla \vec{E} = 0 \end{cases}$$

$\vec{E}$ = electric field vector

$\vec{H}$ = magnetic field vector

c = speed of light

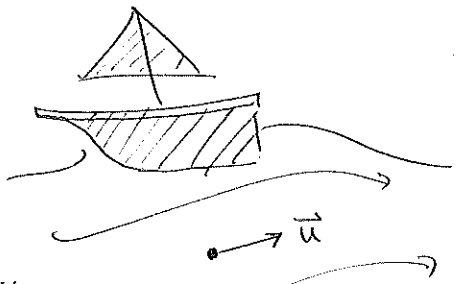
First-order, linear, homog. system for 3D vectors $\vec{H} = \vec{H}(x,y,z)$ and $\vec{E} = \vec{E}(x,y,z)$

note: $\nabla = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)$ and $\nabla \times \vec{H}$ = curl

$\nabla \vec{H}_z$ = gradient.

12) Euler's equations for a compressible fluid

$$\begin{cases} \rho_t + \nabla(\rho \vec{u}) = 0. \\ \vec{u}_t + (\vec{u} \circ \nabla)\vec{u} + \frac{1}{\rho}\nabla p = 0. \end{cases}$$



$\vec{u} = \vec{u}(x,y,z)$ = fluid velocity.

ρ = density

$p = p(\rho)$ fluid pressure.

first-order quasi-linear system for $\vec{u}$ and ρ.

(*) and (* *) :

Definition (Linear and nonlinear PDEs).

(i) PDE is linear if the equation for the suitably differentiable function $u(x_1, x_2, \ldots, x_n)$ is such that u and its derivatives occur linearly, and with coefs. that are functions of the independent variables only.

If the PDE contains a function, $f(x_1, \ldots, x_n)$ that only depends on indep. variables, then the PDE is inhomogeneous. If $f \equiv 0$, then PDE is homogeneous.

(ii) The PDE is called <u>semilinear</u> if all derivs. of u occur linearly (possibly with coefs. that depend on indep. vars.), but u itself occurs nonlinearly.

(iii) A PDE of order k is called <u>quasilinear</u> if its partial derivs. of order k appear linearly, possibley with coefs that are functions of u and derivatives of u of order less than k and also the indep. vars.

<u>Example</u>: Semilinear and quasi-linear first order PDEs:

$$\underbrace{a(x,y)\frac{\partial u}{\partial x} + b(x,y)\frac{\partial u}{\partial y} = f(x,y,u)}_{\text{semi-linear}}$$

$$\underbrace{a(x,y,u)\frac{\partial u}{\partial x} + b(x,y,u)\frac{\partial u}{\partial y} = f(x,y,u)}_{\text{quasi-linear.}}$$

<u>Examples</u>: (Baby PDEs)

(i): $u_{xx} = 0$ where $u = u(x,y)$

$$\Rightarrow u_x = f(y) \Rightarrow u = \underset{\nearrow}{f(y)}x + \underset{\nearrow}{g(y)}$$

no longer constants

(ii) $u_{xx} + u = 0$ where $u = u(x,y)$

This is an "ODE" : $u = C(y)e^{rx}$ (ansatz)

$\Rightarrow$ $u = f(y)\cos x + g(y)\sin x$.

(iii) $u_{xy} = 0$

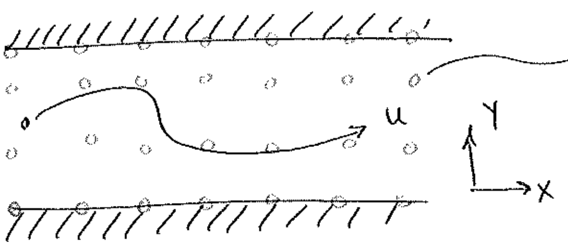
Integrate in x with y fixed $\Rightarrow$ $u_y = f(y)$

" " y " x " $\Rightarrow$ $u = F(y) + G(x)$

where $F'(y) = f(y)$.

Of course these are basically as hard as sloving ODEs like $y' = 0$!

Why are PDEs so hard? One way of thinking: they require so much more data! Consider solving for 2D flow past a ~~strip~~ channel



Each point (x_i, y_i) must record on velocity u_i.

If we split (discretize) the xy-domain into 100×100 pts,

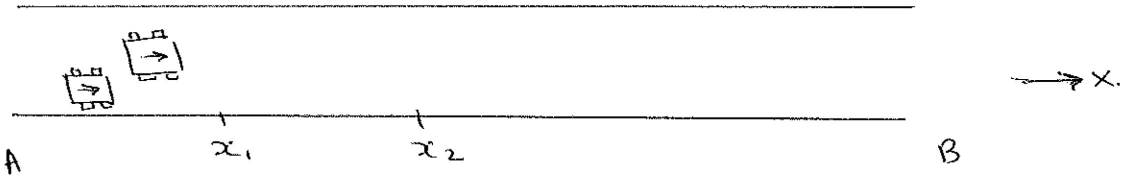
we would need $100^2 = 10,000$ lines of coordinates (x_i, y_i, u_i)

If there is an addition (z) dimension, we would need

$10^3 = 1,000,000$ pts in total!

§ FROM PHYSICS TO PDES

1) Traffic flow (Simple transport)



- Consider a model of flow of traffic on a long straight road between A & B.

- ∃ so many cars that we can assume the flow is represented by a density

  $\rho(x,t)$ = number of cars in a unit length of road at (x,t). (e.g. $\rho = 5 \frac{\text{cars}}{\text{metre}}$)

- An "observer" records a flux of vehicles

  $q(x,t)$ = number of cars at time t passing through x in a unit of time (e.g. $q = 5 \frac{\text{cars}}{\text{sec.}}$)

Number of cars between x_1, x_2 $= \displaystyle\int_{x_1}^{x_2} \rho(x,t) \cdot dx.$

Change in number of cars in $[x_1, x_2]$ $= \displaystyle\frac{d}{dt} \int_{x_1}^{x_2} \rho(x,t) \cdot dx$

- But $\frac{d}{dt}\int_{x_1}^{x_2} \rho(x,t)\cdot dx = \underbrace{q(x_1,t) - q(x_2,t)}_{\text{difference between the number of vehicles entering } x_1 \text{ and leaving } x_2}$

But $q(x_2,t) - q(x_1,t) = \int_{x_1}^{x_2} \frac{\partial}{\partial x} q(x,t)\cdot dx$

by Fund. Thm. of Calc.

$$\Rightarrow \underbrace{\int_{x_1}^{x_2} \left\{ \rho_t(x,t) + q_x(x,t) \right\} \cdot dx = 0.}_{\text{Integral conservation law. [for the cars].}}$$

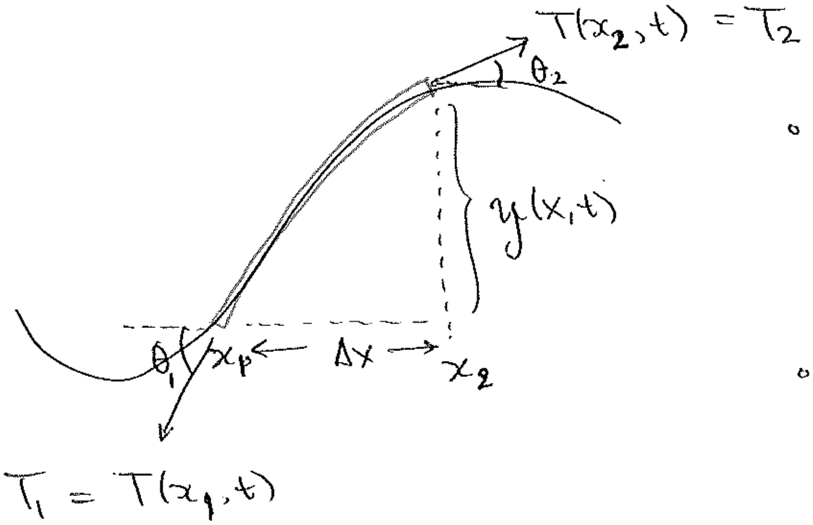
However note that x_1 and x_2 are arbitrary. Whereas the integrand is a function of x. The only way this can happen is if

$$\underbrace{\boxed{\rho_t(x,t) + q_x(x,t) = 0}}_{\text{PDE conservation law for the traffic flow.}}$$

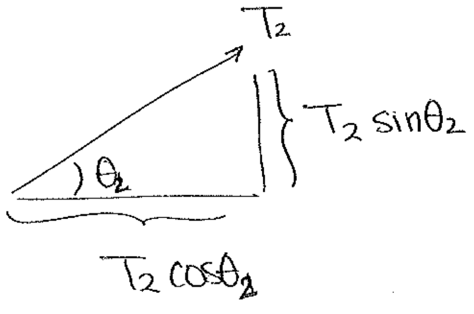
- We need a relationship between density ρ and flux q. Let us assume that all vehicles move right at constant speed, $c>0 \Rightarrow q = c\rho$.

$$\Rightarrow q_x = c\rho_x \Rightarrow \boxed{\rho_t + c\rho_x = 0}$$ ONE-DIM. TRANSPORT EQN

2) Wave Equation for a vibrating string



- String assumes small vibrations., so y^2 and y_x^2 are small
- Motion is entirely transverse (point a distance x_p remains @ x_p $\forall$ t)
- String is at tension T and density of string is constant, ρ.
- Gravity & air resist. negligible.



Balancing forces and Newton's Law:

$$\begin{cases} T_2\cos\theta_2 = T_1\cos\theta_1 \\ T_2\sin\theta_2 - T_1\sin\theta_1 = \underbrace{(\rho\Delta x)}_{\approx \text{mass}} \underbrace{\frac{\partial^2 y}{\partial^2 t^2}(x_0,t)}_{\text{acceleration}} \end{cases}$$

for $x_0 \in [x_1, x_2]$

note that if y_x is small, then,

$T_2\cos\theta_2 \approx T_2$ and $T_1\cos\theta_1 \approx T_1$ indicating that tension is approx. const. over Δx

Divide vert. balance by $T_2 \cos\theta_2 \simeq T_1 \cos\theta_1$

$$\Rightarrow \quad \frac{\tan\theta_2 - \tan\theta_1}{\Delta x} = \frac{\rho}{T} \cdot \frac{\partial^2 y}{\partial t^2}(x_0, t)$$

note $\tan\theta \sim \theta \sim y_x$ when y_x and θ is small.

$$\Rightarrow \frac{\tan\theta_2 - \tan\theta_1}{\Delta x} \sim \frac{y_x(x_2) - y_x(x_1)}{\Delta x} = \frac{\partial^2 y}{\partial x^2}(a, t)$$

for some $a \in [x_1, x_2]$ by MVT.

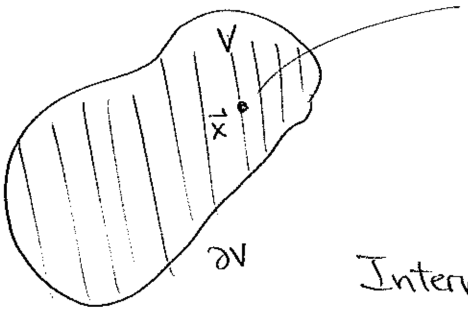
If we now let $\Delta x \to 0$, and assume y_{xx}, y_{tt} are continuous, we get:

$$\frac{\partial^2 y}{\partial t^2} = \left(\sqrt{\frac{T}{\rho}}\right)^2 \cdot \frac{\partial^2 y}{\partial x^2} \Rightarrow \boxed{y_{tt} = c^2 y_{xx}}$$

where $c = \sqrt{\frac{T}{\rho}}$ has units of $\sqrt{\frac{kg \cdot m/s^2}{kg/m}} = \frac{m}{s}$ (velocity)

The equation (in $u(x,t)$) $\boxed{u_{tt} = c^2 u_{xx}}$ is the wave equation.

3) Heat equation



At each point, we assume there is an energy density, $E(\vec{x},t)$ such that

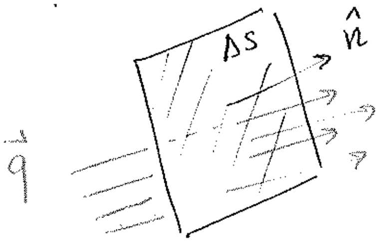
$$\text{Internal heat energy} = \int_V E(\vec{x},t) \cdot dV$$

Volume integral

Additional assumptions :

(i) homogeneous solid (ii) continuous solid.
(iii) isotropic solid (does not change in volume with heat)

We assume there is a heat flux vector, $\vec{q}(\vec{x},t)$ such the amount of heat flow past a (small) ~~volume~~ plane with unit normal $\vec{n}$ is $(\vec{q} \circ \vec{n} \cdot \Delta S)$



Conservation of energy :

$$\underbrace{\frac{d}{dt}\int_V E(\vec{x},t) \cdot dV}_{\text{change in internal heat}} + \underbrace{\int_{\partial V} \vec{q}(\vec{x},t) \circ \hat{n}\, dS}_{\text{amount of heat flow past boundaries}} = 0$$

Divergence theorem $\Rightarrow$ $\displaystyle\int_{\partial V} \vec{q} \circ \vec{n} \cdot dS = \int_V \nabla \circ \vec{q} \cdot dV$

Equating the integrands and since this applies $\forall V$,

$$\frac{\partial E}{\partial t} + \nabla \circ \vec{q} = 0 .$$

↑ another conservation PDE.

We need an (experimental) law which relates $\vec{q}$ with E as well as the temperature. For most materials:

$$E = \underset{\text{density}}{\rho}\, \underset{\text{specific heat}}{c}\, T$$

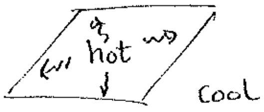
(energy is proportional to temperature)

Also, Fourier's Law states

$$\vec{q} = -\underset{\text{const. (therm. conductivity)}}{k} \nabla T$$

(heat flux is prop. to gradient in temperature)

↑ To remember the sign, recall heat always moves warm ⤳ cold.



Combining these relations:

$$\boxed{\rho c \frac{\partial T}{\partial t} = k \nabla^2 T}$$

<u>heat</u> <u>equation</u>