SERIES SOLUTIONS AROUND REGULAR SINGULAR POINTS

11

From the previous lecture, we learned about series solutions expanded about analytic points. Notice that a similar theorem can be developed for the case of inhomogeneous equations:

Theorem 11.1. *Given the ODE* $y'' + p(x)y' + q(x)y = h(x)$*, if* p*,* q*, and* h *are analytic at* $x = x_0$*, then the general solution,* $y(x)$*, is also analytic at* x_0*, and the series* $y = \sum a_n(x - x_0)^n$ *converges for* $|x - x_0| < R$*, where* $R \geq$ *the distance to the nearest singularity of* p*,* q*, or* h*.*

The proof is similar to the proof of the homogeneous case.

But what happens if x_0 is a singular point? (*i.e.* p or q is not analytic at x_0). Consider for example the equation

$$x^2 y'' - y = 0,$$

at the point $x_0 = 0$. Since $p(x) \equiv 0$ and $q(x) = -1/x^2$, which is not analytic at $x = 0$, then in this case, the relevant point is singular. In fact, this particular equation is a case of an important class of ODEs called **Euler-Cauchy problems**. We have

Definition 11.1 (Euler-Cauchy problem). *Euler-Cauchy equations are of the form*

$$x^2 y'' + axy' + by = 0, \tag{11.1}$$

where a *and* b *are constant.*

To solve Euler-Cauchy problems, we make the substitution $x = e^t$. Then by the chain rule, derivatives become

$$\frac{d}{dx} = \frac{dt}{dx}\frac{d}{dt} = \frac{1}{x}\frac{d}{dt} \tag{11.2}$$

$$\frac{d^2}{dx^2} = \frac{1}{x}\frac{d}{dt}\left(\frac{1}{x}\frac{d}{dt}\right) = \frac{1}{x^2}\frac{d^2}{dt^2} - \frac{1}{x^2}\frac{d}{dt}. \tag{11.3}$$

After simplifying the ODE becomes

$$\frac{d^2y}{dt^2} + (a-1)\frac{dy}{dt} + by = 0,$$

which is simply a second order ODE with constant coefficients, so we know how to solve this. We set $y = e^{rt}$, and solve the equation for r. This leads to three cases:

$$(i) \qquad y_1, y_2 = e^{r_1 t},\ e^{r_2 t} \tag{11.4}$$

$$(ii) \qquad y_1, y_2 = e^{r_1 t},\ e^{r_1 t} + te^{r_1 t} \tag{11.5}$$

$$(iii) \qquad y_1, y_2 = e^{\lambda t}\sin \mu t,\ e^{\lambda t}\cos \mu t. \tag{11.6}$$

Returning the independent variable to x, we thus have for the three cases:

$$(i) \qquad y_1, y_2 = x^{r_1}, x^{r_2} \tag{11.7}$$

$$(ii) \qquad y_1, y_2 = x^{r_1},\ \ln|x| x^{r_1} \tag{11.8}$$

$$(iii) \qquad y_1, y_2 = x^{\lambda} \sin(\mu \ln|x|),\ x^{\lambda} \cos(\mu \ln|x|) \tag{11.9}$$

In practice, the class of Euler-Cauchy problems is important enough that you are not expected to go through the derivation process. Here is an example:

Example 11.1. Find the general solution of

$$x^2 y'' - y = 0.$$

We let $y = x^r$. The indicial equation is

$$r^2 - r - 1 = 0,$$

so that $r = 1/2 \pm \sqrt{5}/2$. Thus the general solution is given by

$$y = c_1 x^{1/2+\sqrt{5}/2} + c_2 x^{1/2-\sqrt{5}/2}.$$

So now what if we instead have an equation like

$$x^2 y'' - (1+x) y = 0. \tag{11.10}$$

Notice that when x is small, then $(1+x) \sim 1$, and the left hand side of the equation can be expected to behave like $x^2 y'' - y$, which is of Euler-Cauchy type. One idea is then to try the series expansion

$$y = x^r \sum_{n=0}^{\infty} a_n x^n, \tag{11.11}$$

so that we have

$$y' = \sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1} \tag{11.12}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2}. \tag{11.13}$$

The ODE becomes

$$\sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r} = \sum_{n=0}^{\infty} a_n x^{n+r} - \sum_{n=0}^{\infty} a_n x^{n+r+1} = 0. \tag{11.14}$$

Shifting the index of the third sum gives

$$[r(r-1)a_0 - a_0]\, x_r + \sum_{n=1}^{\infty} \{[(n+r)(n+r-1) - 1]\, a_n - a_{n-1}\}\, x^{n+r} = 0.$$

The coefficient of x^r gives either $a_0 = 0$, which causes $a_1 = a_2 = \ldots = 0$, producing the trivial solution. Otherwise, we are led to the indicial equation

$$r^2 - r - 1 = 0,$$

and we recover the result of $r = 1/2 \pm \sqrt{5}/2$ as before. For $n \geq 1$, the coefficient of x^{n+r} gives the recurrence relation

$$a_n = \frac{a_{n-1}}{(n+r)(n+r-1)}. \tag{11.15}$$

We may then use $r = r_1$ or $r = r_2$ to produce two independent solutions.

However, our process leads to three issues:

1. Does this form of series $y(x) = \sum a_n x^{n+r}$ always work for any singular point? (No! The point must be regular)
2. What if r is repeated?
3. What if r_1 and r_2 differ by an integer?

Definition 11.2. *Given the ODE $y'' + p(x)y' + q(x)y = 0$, where $x = x_0$ is a singular point, we say that x_0 is a **regular singular point** if the functions $(x - x_0)p(x)$ and $(x - x_0)^2 q(x)$ are analytic at $x = x_0$. Otherwise, we say $x_0)$ is an **irregular singular point**.*

What we need is a condition that tells us whenever an equation is locally like an Euler-Cauchy problem. Suppose we have a differential equation of the form $A(x)y'' + P(x)y' + Q(x)y = 0$, then this equation will be locally similar to the Euler-Cauchy problem at $x = x_0$ if $A(x) \sim (x - x_0)^2$, $P(x) \sim (x - x_0)$, and $Q(x) \sim$ constant.

Example 11.2. Consider the problem $x^2 y'' - (1 - x)y = 0$. Then

$$p(x) = 0 \qquad \text{and} \qquad q(x) = \frac{1 + x}{-x^2}.$$

Clearly, q is not analytic at $x = 0$. However, note that $x^2 q = -(1 + x)$ is analytic at $x = 0$, so the point is a regular singular point.

Example 11.3. Consider the problem $x^3 y'' - (1 + x)y = 0$. In this case,

$$p(x) = 0 \qquad \text{and} \qquad q(x) = -\frac{(1 + x)}{x^3},$$

so q is still not analytic at $x = 0$. In this case, $x^2 q = -\frac{(1+x)}{x}$, which is also not analytic at $x = 0$. Thus, in this case, $x = 0$ is an irregular singular point. Alternatively, we could have seen that this equation behaves like

$$x^3 y'' - y = 0,$$

near $x = 0$, which is clearly not Euler-Cauchy.

The important now follows:

Theorem 11.2 (First Frobenius Theorem). *Let $x = x_0$ be a regular singular point of the ODE $y'' + p(x)y' + q(x)y = 0$. Then, the general solution is composed of two independent series solutions:*

$$y_1 = |x - x_0|^{r_1} \sum_{n=0}^{\infty} a_n^{(1)} (x - x_0)^n \tag{11.16}$$

$$y_2 = |x - x_0|^{r_2} \sum_{n=0}^{\infty} a_n^{(2)} (x - x_0)^n, \tag{11.17}$$

where r_1 and r_2 are the roots of the indicial equation

$$r(r-1) + p_0 r + q_0 = 0,$$

and are such that $r_1 \neq r_2$, and $r_1 - r_2 \neq$ integer. Here

$$p_0 = \lim_{x \to x_0} (x - x_0) p(x) \qquad \textit{and} \qquad q_0 = \lim_{x \to x_0} (x - x_0)^2 q(x).$$

These series solutions then converge for $|x - x_0| < R$, where $R \geq$ the distance to the nearest singularity of $(x - x_0)p(x)$ and $(x - x_0)^2 q(x)$.

Proof. Without loss of generality, we can assume that $x_0 = 0$. Then, we assume that we have the two series

$$xp(x) = \sum p_n x^n \qquad \text{for } |x| < R_1 \tag{11.18}$$

$$x^2 q(x) = \sum q_n x^n \qquad \text{for } |x| < R_1 \tag{11.19}$$

We then write the ODE

$$y'' + p(x)y' + q(x)y = 0 \Rightarrow x^2 y'' + x[xp(x)]y' + [x^2 q(x)]y = 0.$$

Thus, if we take

$$y = |x|^r \sum_{n=0}^{\infty} a_n x^n, \qquad \text{for } |x| < R \tag{11.20}$$

Again, without loss of generality, we shall assume that $x > 0$ (in order to drop the absolute value notation). Substituting the series into the ODE gives

$$\sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r} + \left(\sum_{n=0}^{\infty} p_n x^n \right) \left(\sum_{m=0}^{\infty} (m+r) a_m x^{m+r} \right) + \left(\sum_{n=0}^{\infty} q_n x^n \right) \left(\sum_{m=0}^{\infty} (m+r) a_m x^{m+r} \right) = 0. \tag{11.21}$$

Using the usual property of convolutions gives

$$\sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r} + \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \left[p_{n-m} (n+r) a_m \right] x^{n+r} + \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \left[q_{n-m} (n+r) a_m \right] x^{n+r} = 0. \tag{11.22}$$

If we then isolate the coefficients of x^{n+r}, we get

$$\underbrace{[r(-1)q_0 + (p_0 r + q_0)a_0]}_{F(r)} x^r + \sum_{n=1}^{\infty}[(n+r)(n+r-1)a_n + \sum_{m=0}^{n}[p_{n-m}(m+r) + q_{n-m}]a_m]x^{n+r} = 0 \quad (11.23)$$

We must pick r such that $F(r) = r(r-1) + p_0 r + q_0 = 0$, otherwise

$$a_0 = 0 \Rightarrow a_1 = 0 \Rightarrow ... \Rightarrow a_n = 0 \Rightarrow \text{trivial solution} \quad (11.24)$$

Thus, the equation $F(r) = 0$ gives two roots r_1 and r_2. The subsequent terms, with x^{n+r}, for $n \geq 1$ give

$$(\underbrace{(n+r)(n+r-1) + p_0(n+r)}_{F(n+r)} + q_0)a_n + \sum_{m=0}^{n-1}((m+r)p_{n-m} + q_{n-m})a_m = 0 \quad (11.25)$$

so we conclude that

$$a_n = \frac{-1}{F(n+r)} \sum_{m=0}^{n-1}((m+r)p_{n-m} + q_{n-m})a_m \quad (11.26)$$

We can plug in the values $r = r_1$ and $r = r_2$ to get sequences $a_n^{(1)}$ and $a_n^{(2)}$ respectively. It is important to note that for the sequences a_n to exist and the solutions to be linearly independent, we need $r_1 \neq r_2$ (so the sequences are distinct), and $F(n+r_1) \neq 0$ and $F(n+r_2) \neq 0$ for all $n \geq 1$. In other words, we cannot have $r_1 - r_2 = M \in \mathbb{Z}$, since then $F(M+r_2) = F(r_1) = 0$. Thus, given that $r_1 - r_2 \neq M$, then we know that $F(n+r_1) \neq 0$ and $F(n+r_1) \neq 0$ for all $n \geq 1$. Since $F(r)$ is quadratic $\Rightarrow r_1, r_2$ are the only roots.

We now need to show that such series solutions are convergent. The coefficients satisfy:

$$F(n+r)a_n = -\sum_{m=0}^{n-1}((m+r)p_{n-m} + q_{n-m})\, a_m$$
$$\Rightarrow |F(n+r)|\,|a_n| \leq \sum_{m=0}^{n-1}((m+r)p_{n-m} + q_{n-m})\,|a_m|\,. \quad (11.27)$$

Note that $\sum p_n x^n$ and $\sum q_n x^n$ converge for $|x| < \min\{R_1, R_2\} = R$. Thus, if we pick $|x| = S < R$, then there exists $k > 0$ such that $|p_n S^n| \leq k$ and $|q_n S^n| \leq k$ for any n. We thus bound

$$|p_n|, |q_n| \leq \frac{k}{S^n},$$

and we can write

$$|F(n+r)|\,|a_n| \leq \sum_{m=0}^{n-1} [|m+r|+1]\,\frac{k}{S^{n-m}}|a_m|$$

We can now define a new sequence, b_n, such that $b_0 = |a_0|$,

$$|F(n+r)|b_n = \frac{k}{S^n}\sum_{m=0}^{n-1}[|m+r|+1]\,b_m S^m,$$

for all $n \geq 1$. And so, we can show by induction that $|a_n| \leq b_n \forall n \geq 1$ (exercise).

$$|F(n+r+1)|\,b_{n+1} = \frac{k}{S^{n+1}}\sum_{m=0}^{n}[|m+r|+1]\,b_m S^m \tag{11.28}$$

$$= \frac{1}{S}\left(\underbrace{\frac{k}{S^n}\sum_{m=0}^{n-1}[|m+r|+1]\,b_m S^m}_{F|(n+r|b_n} + \frac{k}{S^n}[|n+r|+1]\,b_n S^n\right) \tag{11.29}$$

$$S|F(n+r+1)|\,b_{n+1} = |F(n+r)|\,b_n + k[|n+r|+1]\,b_n \frac{b_{n+1}}{b_n} \tag{11.30}$$

$$= \frac{|F(n+r)|\,b_n + k[|n+r|+1]}{S|F(n+r+1)|}, \tag{11.31}$$

and

$$\frac{1}{S}\left|\frac{b_{n+1}}{b_n}\frac{x^{n+1}}{x^n}\right| \to \frac{|x|}{S},$$

as $n \to \infty$. Thus we need $|x| < S$ for $\sum b_n x^n$ to converge. Since $S < R$ is arbitrary, we have that $\sum b_n x^n$ converges for $|x| < R$, and thus $\sum a_n x^n$ converges absolutely for $|x| < R$ (so for $0 < x < R$), and so does $\sum a_n x^{n+r}$ (except possibly at $x = 0$ where there may be a singularity).

Note: for $x < 0$ the series is really just $y = |x^r|\sum_{n=0}^{\infty} a_n x^n$ If $x < 0$, then

$$y = (-x)^r \sum a_n x^n \tag{11.32}$$

$$y' = (-x)^r \sum (n+r) a_n x^{n-1} \tag{11.33}$$

$$y'' = (-x)^r \sum (n+r)(n+r-1) a_n x^{n-2} \tag{11.34}$$

This results in the same recurrence for a_n, and all of the discussion above remains valid for $-R < x < 0$, with the series converging for $|x| < R$, where $R \geq \min\{R_1, R_2\}$. □