SERIES SOLUTIONS AROUND ORDINARY POINTS

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Up until now, all the ODEs we have studied have been expressible in terms of elementary functions, *e.g.* $\cos x$, $\sin x$, e^x, $\log x$, x^n, etc. In practice, however, most differential equations which arise in real problems cannot be solved in closed form. For example, there is no closed form solution for the relatively 'simple' second order linear equation

$$y'' + p(x)y' + q(x)y = h(x),$$

and the next best thing is to write down *series approximations*. Concerning the topic of series approximations, there are two key questions: (i) how do we know *a priori* what form the series takes? (ii) Is the series convergent, and if so, how far?

Consider the **Airy Equation**:

$$y'' = xy. \tag{10.1}$$

Let us guess that the solution has a Taylor series expansion around $x = 0$. We write

$$y(x) = \sum_{n=0}^{\infty} a_n x^n.$$

Substituting this form into the ODE gives

$$\sum_{n=0}^{\infty} a_n n(n-1)x^{n-2} = \sum_{n=0}^{\infty} a_n x^{n+1}. \tag{10.2}$$

Examining the left sum, notice that

$$\sum_{n=0}^{\infty} a_n n(n-1)x^{n-2} = \sum_{n=2}^{\infty} a_n n(n-1)x^{n-2} = 2a_2 + \sum_{n=3}^{\infty} a_n n(n-1)x^{n-2},$$

and we can shift the dummy index, $n - 2 = m + 1$ to give

$$2a_2 + \sum_{m=0}^{\infty} a_{m+3}(m+3)(m+2)x^{m+1} = 2a_2 + \sum_{n=0}^{\infty} a_{n+3}(n+3)(n+2)x^{n+1}, \tag{10.3}$$

where in the right side, we have simply returned the dummy index back to n. Combining (10.2) with (10.3) gives after simplifying

$$2a_2 + \sum_{n=0}^{\infty} [(n+3)(n+2)a_{n+3} - a_n]x^{n+1} = 0.$$

Equating coefficients, we get

$$a_2 = 0,$$

and also

$$(n+3)(n+2)a_{n+3} - a_n = 0,$$

for $n = 0, 1, 2, \ldots$. Thus

$$a_{n+3} = \frac{a_n}{(n+3)(n+2)}.$$

This is known as a **recurrence relation** (or difference equation). Note that since $a_2 = 0$, then from the recurrence relation, we also have $a_5 = 0$, $a_8 = 0$, and so on. Thus

$$a_{3n+2} = 0,$$

for $n = 0, 1, 2, \ldots$. Assuming that $a_0 \neq 0$, then the first nontrivial coefficient is a_3. From a little computation, we have

$$a_3 = \frac{a_0}{3 \cdot 2} \tag{10.4}$$

$$a_6 = \frac{4 \cdot 1}{6!} a_0 \tag{10.5}$$

$$a_9 = \frac{7 \cdot 4 \cdot 1}{9!} a_0 \tag{10.6}$$

$$\ldots = \ldots \tag{10.7}$$

$$a_{3n} = \frac{(3n-2)(3n-5)\ldots 2 \cdot 1}{(3n)!} a_0. \tag{10.8}$$

There is a similar sequence of coefficients stemming from a_1. This gives

$$a_4 = \frac{1}{4 \cdot 3} a_1 \tag{10.9}$$

$$a_7 = \frac{5 \cdot 2}{7!} a_1 \tag{10.10}$$

$$\ldots = \ldots \tag{10.11}$$

$$a_{3n+1} = \frac{(3n-1)(3n-4)\cdots 2}{(3n+1)!} a_1. \tag{10.12}$$

Putting everything together, we are left with

$$y(x) = a_0 \left[\sum_{n=0}^{\infty} \frac{(3n-2)(3n-5)\cdots(1)}{(3n)!} x^{3n} \right] + a_1 \left[\sum_{n=0}^{\infty} \frac{(3n-1)(3n-4)\cdots 2}{(3n+1)!} x^{3n+1} \right], \tag{10.13}$$

i.e. two independent solutions with associated coefficients, a_0 and a_1, which are determined by initial (or boundary) conditions. Do the two series converge? Recall the **ratio test**: the power series $\sum a_n x^n$ converges provided

$$\lim_{n \to \infty} \frac{|a_{n+1} x^{n+1}|}{|a_n x^n|} < 1.$$

If we apply to the first series, we get

$$\lim_{n \to \infty} \frac{\left| \frac{(3n+1)(3n-2)\ldots 1}{(3n+1)!} x^{3(n+1)} \right|}{\left| \frac{(3n-2)(3n-5)\ldots 1}{(3n)!} x^{3n} \right|} < 1$$

or

$$|x^3| < \lim_{n\to\infty}(3n+3)(3n+2) \to \infty,$$

which applies for all x fixed. The similar conclusion can be drawn for the second series, and we thus conclude that the general solution converges for all $x \in \mathbb{R}$.

Notice, however, that this procedure does not always work. Consider

$$x^2y'' - y = 0, \tag{10.14}$$

and make the substitution

$$y = \sum_{n=0}^{\infty} a_n x^n.$$

After some simplifying, this gives

$$\sum_{n=0}^{\infty}\Big[n(n-1)-1\Big]a_n x^n = 0,$$

or simply that

$$(n^2-n-1)a_n = 0.$$

We are thus forced to conclude that $a_n = 0$ for all $n = 0, 1, 2, \ldots$. The series solution fails and we get the trivial solution, $y = 0$. Why did this fail? It failed because we used the wrong ansatz for the solution.

Definition 10.1 (Analytic). *A function, $f(x)$, is **analytic** at a point, x_0, if there exists $R > 0$ such that the Taylor Series of f about x_0 converges to $f(x)$ for all $|x - x_0| < R$. We then write*

$$f(x) = \sum_{n=0}^{\infty} a_n (x-x_0)^n,$$

where $a_n = f^{(n)}(x_0)/n!$.

Being analytic is *almost* to being infinitely differentiable…but not quite! In fact, being analytic implies that the function is infinitely differentiable in a neighbourhood of the point, x_0. However, there are some (non-pathological) functions that are infinitely differentiable, but not analytic.

Example 10.1. Consider the three functions

$$f_1(x) = \frac{1}{1-x}$$
$$f_2(x) = \sqrt{x}$$
$$f_3(x) = \frac{\sin x}{x},$$

near the point $x_0 = 0$. We have for the first,

$$f_1(x) = 1 + x + x^2 + \ldots = \sum_{n=0}^{\infty} x^n,$$

and f_1 is analytic at x_0 with $R = 1$. Then for the second, notice that $f_2'(x) = 1/(2\sqrt{x})$, so the derivative is not defined at $x_0 = 0$. Thus, the function is not analytic. Lastly, for the third, note that

$$f_3(x) = 1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \ldots,$$

which is convergent with $R \to \infty$. Thus, f_3 is analytic at $x = 0$.

The next theorem is from complex variables, and shows how the radius of convergence can be determined.

Theorem 10.1 (Radius of convergence). *If $f(x)$ is analytic at x_0 and f has singularities (points where it is not analytic) at points x_1, x_2, $\ldots x_n$, which are complex-valued in general, then the series,*

$$f(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n,$$

converges for $(x - x_0) < R$ where $R \geq \min_j |x_0 - x_j|$. That is, the series will converge at least as far as the nearest singularity.

Example 10.2. Consider $f(x) = 1/(1-x)$. The function is not analytic at $x_1 = 1$, so for the series centered at $x_0 = 0$, we have $R \geq 1$. Similarly, consider $f(x) = 1/(1+x^2)$, which has singularities at $x = \pm i$. We can then check directly from the expansion,

$$\frac{1}{1+x^2} = 1 - x^2 + x^4 - \ldots,$$

that $R = 1$.

Let us now return to the usual second order linear ODE:

$$y'' + p(x)y' + q(x)y = 0 \tag{10.15}$$

Definition 10.2 (Ordinary and singular points). *If at $x = x_0$, both $p(x)$ and $q(x)$ are analytic, then x_0 is an* ***ordinary point*** *of the ODE. Otherwise, x_0 is called a* ***singular point****.*

We now come to the important theorem of this lecture, which concerns series solutions expanded around ordinary points.

Theorem 10.2 (Series solutions at ordinary points). *Let $x = x_0$ be an ordinary point of the ODE (10.15). Then the general solution, $y(x)$, is analytic at x_0, and the associated series has a radius of convergence $R \geq |x_0 - x_1|$, where x_1 is the nearest point of non-analyticity of p and q to x_0. If neither p and q have singularities, then $R \to \infty$.*

We will go through some of the main ideas of the proof in class.