1. Non-dimensionalizing

The dimensional Navier-Stokes equations are

$$\rho\left(\mathbf{u}_t + (\mathbf{u}\cdot\nabla)\mathbf{u}\right) = -\nabla p + \mu\nabla^2\mathbf{u} \tag{1}$$

$$\nabla\cdot\mathbf{u} = 0 \tag{2}$$

By non-dimensionalizing according to

$$x, y, z \sim O(L) \qquad u, v, w \sim O(U)$$
$$t \sim O(L/U) \qquad p \sim O([p]),$$
$$\mathrm{Re} = UL/\nu$$

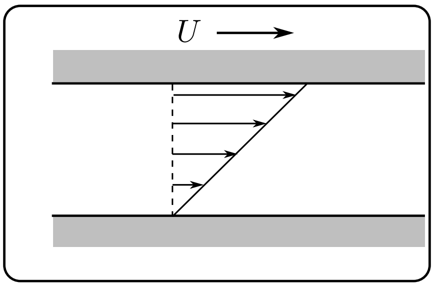
obtain the standard dimensional Navier-Stokes equations

$$\left(\mathbf{u}_t + (\mathbf{u}\cdot\nabla)\mathbf{u}\right) = -\frac{[p]L}{\rho U^2}\nabla p + \frac{1}{\mathrm{Re}}\nabla^2\mathbf{u} \tag{3}$$

$$\nabla\cdot\mathbf{u} = 0, \tag{4}$$

2. **Match Me!**: Match the picture on the left to its leading-order governing equation on the right (all the equations, except for IV, are non-dimensional). Then for each case, prove the corresponding result.

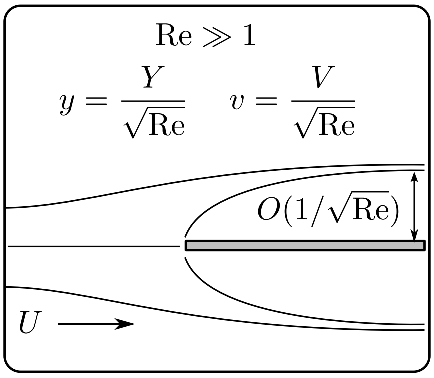
A) Steady Couette Flow



I

$$\nabla^2 \mathbf{u} = \nabla p$$
$$\nabla \cdot \mathbf{u} = 0$$

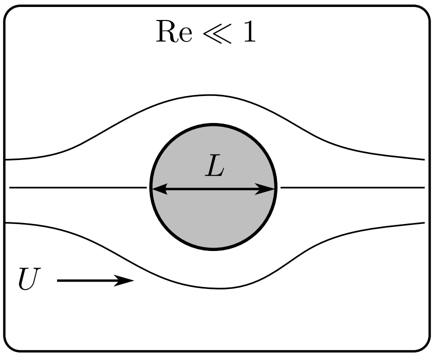
B) Prandtl's Boundary Layer



II

$$0 = -p_x + u_{yy}$$
$$0 = -p_y$$
$$0 = u_x + v_y$$

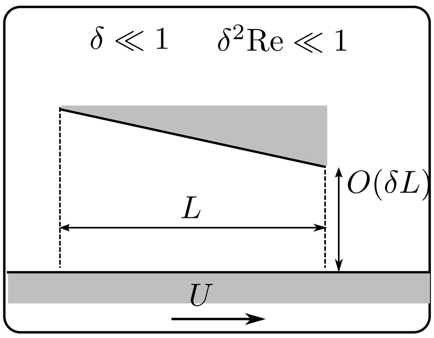
C) Slow Flow Equations



III

$$u u_x + V u_Y = -p_x + u_{YY}$$
$$0 = -p_Y$$
$$u_x + V_Y = 0$$

D) Slider Bearing (Lubrication Theory)



IV

$$0 = -p_x + \mu u_{yy}$$
$$0 = -p_y$$

Solution:

(a) Goes with IV.

Let $\mathbf{u} = [u(y), 0, 0]$ and it follows easily.

(b) Goes with III.

Start with the usual non-dimensional equations (3)-(4). Choose $[p] = \rho U^2$ to balance inertia and pressure. Now let $y = Y/\sqrt{\text{Re}}$ and $v = V/\sqrt{\text{Re}}$ and the result follows.

(c) Goes with I.

Start with the usual non-dimensional equations (3)-(4). Choose $[p] = \rho U^2/\text{Re} = \mu U/L$ to balance pressure and viscosity.

(d) Goes with II.

It's a little bit easier to simply start from the 'standard' non-dimensionalized equations,

$$(\mathbf{u}_t + (\mathbf{u} \cdot \nabla)\mathbf{u}) = -\frac{[p]L}{\rho U^2}\nabla p + \frac{1}{\text{Re}}\nabla^2\mathbf{u}$$

For lubrication flows, there is an 'extra' scaling requiring $y = \delta y'$ and $v = \delta v'$ (It's a lot like the high Re flow near a plate!). We substitute this extra scaling and drop primes to get

$$\begin{aligned} \delta^2\text{Re}\left(u_t + uu_x + \delta v u_y\right) &= -\frac{\delta^2 L^2 [p]}{\mu U} p_x + \delta^2 u_{xx} + u_{yy} && (5) \\ \delta^2\text{Re}\left(v_t + uv_x + \delta v v_y\right) &= -\frac{L^2 [p]}{\mu U} p_y + \delta v_{xx} + v_{yy} && (6) \\ u_x + v_y &= 0 && (7) \end{aligned}$$

The correct balance requires $[p] = \mu U/\delta^2 L^2$. Then keep only the $O(1)$ terms.

3. **Slider bearing (Q4, 2008 B6a exam)**

(a) A viscous fluid fills a 2D slider bearing of length L between the surfaces $y = 0$ and $y = h_0 h(x/L)$ where $h(0) = 1$. The wall $y = 0$ is moving with constant velocity U in the x direction and the pressure in the fluid at $x = 0$ and $x = L$ is a constant p_0. If $h_0/L, Uh_0^2/\nu L \ll 1$, show that the pressure p satisfies the equation

$$\frac{d}{dx}\left[h^3\frac{dp}{dx} - 6h\right] = 0,$$

where both p and x are suitable nondimensional variables.

(b) If $h(x) = e^{-\alpha x}$, show that if $\alpha > 0$, the pressure within the bearing first increases with real x and then decreases, and that the maximum pressure occurs at the point,

$$x = x_m = -\frac{1}{\alpha}\log\left(\frac{3}{2}\frac{e^{2\alpha}-1}{e^{3\alpha}-1}\right)$$

Solution:

(a) In order to relate this to Q1, we let $\delta = h_0/L$. Then indeed, $\delta = h_0/L \ll 1$ and $\delta^2\mathrm{Re} = h_0^2 U/\nu L \ll 1$ and thus

$$\frac{\partial^2 u}{\partial y^2} = 0, \qquad \frac{\partial p}{\partial y} = 0, \qquad \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

$$u = 1, v = 0 \quad \text{on } y = 0$$

$$u = 0, v = 0 \quad \text{on } y = h(x)$$

First, we cross-layer average the continuity equation,

$$\int_0^h \frac{\partial u}{\partial x}\,dy + \int_0^h \frac{\partial v}{\partial y}\,dy = \frac{\partial}{\partial x}\int_0^h u\,dy - u\frac{\partial h}{\partial x}\bigg|_{y=0}^{y=h} + v\big|_{y=0}^{y=h} = \frac{\partial}{\partial x}(h\overline{u}) = 0$$

Now we get the expressions for $\overline{u}$. By the y momentum equation, $p = p(x)$. Then we integrate x momentum equation to get

$$u(x,y) = \frac{p_x}{2}y^2 + C(x)y + D(x)$$

Applying the boundary conditions gives

$$u(x,y) = \frac{p_x}{2}y(y-h) + \left(1 - \frac{y}{h}\right)$$

Computing the average gives

$$\overline{u} = \frac{1}{12}(p_x h^2 + 6)$$

Combining gives us the result.

(b) Plugging in $h(x) = e^{-\alpha x}$ and solving for $p(x)$ gives,

$$p(x) = \frac{3}{\alpha}e^{2\alpha x} + \frac{C}{3\alpha}e^{3\alpha x} + D$$

where C and D are constants. We know that $p(0) = p(L) = p_0 = 0$ (WLOG). Now we are left to solve the system,

$$\begin{aligned} \frac{3}{\alpha} + \frac{C}{3\alpha} + D &= 0 \\ \frac{3}{\alpha}(1 - e^{2\alpha L}) + \frac{C}{3\alpha}e^{3\alpha L} + D &= 0 \end{aligned}$$

and it gives,

$$C = -9\left(\frac{e^{2\alpha L} - 1}{e^{3\alpha L} - 1}\right)$$

Thus,

$$\frac{\partial p}{\partial x} = 6e^{2\alpha x} + Ce^{3\alpha x}$$

Though it's not quite straightforward to show (because there are different regimes in (α, L) space to consider), p_x begins positive, passes through zero, and goes negative. The zero of p_x (and thus the maximum of $p(x)$) is at

$$x = x_m = -\frac{1}{\alpha}\log\left(\frac{3}{2}\frac{e^{2\alpha} - 1}{e^{3\alpha} - 1}\right)$$

4. **Thin film down a ramp (Q2, 2004 B6a exam)**

A 2D viscous fluid layer flows under the action of gravity on a flat plate inclined at an angle δ to the horizontal. Show that, written in terms of coordinates (x tangential to the plate), in the usual slow-scalings, the Navier-Stokes equations are

$$\begin{aligned} \text{Re}\,(\mathbf{u}_t + (\mathbf{u}\cdot\nabla)\mathbf{u}) &= -\nabla p + \nabla^2\mathbf{u} - \left(\frac{\rho g L^2}{\mu U}\right)(\sin\delta, \cos\delta) \\ \nabla\cdot\mathbf{u} &= 0 \end{aligned}$$

where U is the velocity scale, L is the length scale, and Re is the Reynolds number, which you should define.

Now suppose that the (dimensional) thickness of the fluid layer is $y = \delta L h(x,t)$, and that $\delta \ll 1$. Show that in the lubrication approximation, the correct choice of the velocity scale is $U = \rho g L^2 \delta^3/\mu$ and that

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \qquad 0 = -\frac{\partial p}{\partial x} + \frac{\partial^2 u}{\partial y^2} - 1, \qquad 0 = -\frac{\partial p}{\partial y} - 1.$$

Tthe boundary conditions on $y = 0$ are $u = v = 0$, while at $y = h(x, t)$,

$$v = \frac{\partial h}{\partial t} + u\frac{\partial h}{\partial x} \qquad \text{and} \qquad p = \frac{\partial u}{\partial y} = 0.$$

Hence show that

$$\frac{\partial h}{\partial t} = \frac{\partial}{\partial x}\left\{\frac{h^3}{3}\left(1 + \frac{\partial h}{\partial x}\right)\right\}$$

Bonus: How is this question different from the scenario in which the ramp is completely horizontal?

Solution: I found this question a bit awkwardly posed because different scalings on the pressure are needed for the slow-flow and lubrication approximations. Thus p in the first set of formulae is not the same as in the second set!

Nevertheless, the force down the ramp is $\mathbf{F} = -g[\sin\delta, \cos\delta]$. As in Q1, we start with the usual non-dimensional Equations (3)-(4), except now we need to include a forcing term,

$$\begin{aligned} \left(\mathbf{u}_t + (\mathbf{u}\cdot\nabla)\mathbf{u}\right) &= -\frac{[p]L}{\rho U^2}\nabla p + \frac{1}{\text{Re}}\nabla^2\mathbf{u} - \left(\frac{Lg}{U^2}\right)[\sin\delta, \cos\delta] \qquad (8) \\ \nabla\cdot\mathbf{u} &= 0, \qquad (9) \end{aligned}$$

The factor in front of the gravitational forcing is from the scalings of the inertial term. Choose $[p] = \rho U^2/\text{Re} = \mu U/L$ to balance pressure and viscosity and multiply through by Re to get the result.

$$\begin{aligned} \text{Re}\left(\mathbf{u}_t + (\mathbf{u}\cdot\nabla)\mathbf{u}\right) &= -\nabla p + \nabla^2\mathbf{u} - \left(\frac{\rho g L^2}{\mu U}\right)[\sin\delta, \cos\delta] \\ \nabla\cdot\mathbf{u} &= 0 \end{aligned}$$

Now the dimensional thickness of the fluid layer is $y = \delta L h(x, t)$, so we need to re-scale y by δ. Let us simply send $y \mapsto \delta y$. To balance the continuity equation, we will also need to send $v \mapsto \delta v$. Before we balance, note that if $\delta \ll 1$, we expect $[\sin\delta, \cos\delta] \sim [\delta, 1]$. Now the momentum equations becomes

$$\begin{aligned} \text{Re}\left(u_t + uu_x + vu_y\right) &= -p_x + u_{xx} + \frac{1}{\delta^2}u_{yy} - \left(\frac{\rho g L^2}{\mu U}\right)\delta \\ \delta\text{Re}\left(v_t + uv_x + vv_y\right) &= -\frac{1}{\delta}p_y + \delta v_{xx} + \frac{1}{\delta}v_{yy} - \left(\frac{\rho g L^2}{\mu U}\right) \end{aligned}$$

The tricky bit is to realize that you need to also rescale the pressure $p = p/\delta^2$ in order to re-balance the pressure and viscous terms. Doing so we see that the we need to also choose $U = \rho g L^2\delta^3/\mu$ to bring in the gravitational terms. Finally, we arrive at the correct set:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \qquad 0 = -\frac{\partial p}{\partial x} + \frac{\partial^2 u}{\partial y^2} - 1, \qquad 0 = -\frac{\partial p}{\partial y} - 1,$$

The boundary conditions at $y = 0$ are easy: $u = v = 0$. The boundary equations on the free surface include the kinematic and dynamic conditions.

Kinematic: Fluid particles on the surface stay on the surface.

$$\begin{aligned} & \frac{D}{Dt}(h(x,t) - y) = 0 \\ \Rightarrow \quad & \frac{\partial h}{\partial t} + (\mathbf{u} \cdot \nabla)(h(x,t) - y) = 0 \\ \Rightarrow \quad & \frac{\partial h}{\partial t} + u\frac{\partial h}{\partial x} = v \qquad \text{on } y = h(x,t) \end{aligned}$$

Dynamic: Zero stress at the free surface.

We can write the non-dimensional force as

$$\begin{aligned} \sigma &= \frac{\mu U}{L}\begin{bmatrix} -\frac{1}{\delta^2}p + 2\left(\frac{\partial u}{\partial x}\right) & \frac{1}{\delta}\frac{\partial u}{\partial y} + \frac{\partial u}{\partial x} \\ \frac{1}{\delta}\frac{\partial u}{\partial y} + \frac{\partial u}{\partial x} & -\frac{1}{\delta^2}p + \frac{2}{\delta}\left(\frac{\partial u}{\partial y}\right) \end{bmatrix} \\ &\sim \frac{\mu U}{L}\begin{bmatrix} -\frac{1}{\delta^2}p & \frac{1}{\delta}\frac{\partial u}{\partial y} \\ \frac{1}{\delta}\frac{\partial u}{\partial y} & -\frac{1}{\delta^2}p + \frac{2}{\delta}\left(\frac{\partial u}{\partial y}\right) \end{bmatrix} \end{aligned}$$

Setting the stress to zero, we see that,

$$p = \frac{\partial u}{\partial y} = 0 \text{ on } y = h(x,t)$$

Now to derive Reynold's equation, we need to integrate average the continuity equation

$$\begin{aligned} & \int_0^h \frac{\partial u}{\partial x}\, dy + \int_0^h \frac{\partial v}{\partial y}\, dy = 0 \\ \Rightarrow \quad & \frac{\partial}{\partial x}\int_0^h \frac{\partial u}{\partial} dy - u\frac{\partial h}{\partial x} + v\Big|_0^h\, dy = 0 \\ \Rightarrow \quad & \frac{\partial}{\partial x}(h\overline{u}) - u\frac{\partial h}{\partial x} + \frac{\partial h}{\partial t} + u\frac{\partial h}{\partial x} = 0 \end{aligned}$$

where in the second line, don't forget to use Leibniz's Rule(!) and in the third line, we've used the boundary conditions at $y = 0$ and $y = h$. Now to get $\overline{u}$, the y momentum equation gives us,

$$p(x,y) = y - h(x),$$

using the boundary conditions. Whereas after integration, the x momentum equation gives us,

$$u(x,t) = \left(\frac{\partial h}{\partial x} + 1\right)\left[\frac{y^2}{2} - hy\right] \Rightarrow \overline{u} = \frac{1}{3}\left(\frac{\partial h}{\partial x} + 1\right)h^3$$

Thus

$$\frac{\partial h}{\partial t} = \frac{\partial}{\partial x}\left\{\frac{h^3}{3}\left(1 + \frac{\partial h}{\partial x}\right)\right\}$$

as required.