1. PS1: **Deriving Navier Stokes**

   Derive the Navier-Stokes equations (continuity and momentum) for an incompressible, Newtonian, and viscous fluid with body force **g**. Write down what it means for the fluid to be incompressible in terms of the density, ρ. You can use the standard lemmas (*e.g.* Reynold's Transport Theorem, material derivative, etc.), but clearly quote anything you use.

   Solution: Standard.

2. PS2, Q6: **Practise non-dimensionlising equations**

   Start from the **dimensional** Navier-Stokes equations you derived in the last question. Consider **two-dimensional** flow past an obstacle of typical size L, with speed U far away. Derive the non-dimensional equations in terms of the Reynolds number, $\mathrm{Re} = UL/\nu$ for,

   (a) $\mathrm{Re} \gg 1$ leading to inviscid flow

   (b) $\mathrm{Re} \ll 1$ leading to slow flow

   For each above case, you will have to select the correct choice of typical pressure p. Also for each case, explain what the *leading-order* behaviour should be.

   Solution: Let $x = Lx'$, $y = Ly'$, $u = Uu'$, $v = Uv'$, $t = (L/U)t'$. We'll work with the x-momentum equation, but the others are identical.

$$\rho\left(\frac{U}{L/U}u'_{t'} + \frac{U^2}{L}u'u'_{x'} + \frac{U^2}{L}v'u'_{y'}\right) = -\frac{[p]}{L}p'_{x'} + \mu\left(\frac{U}{L^2}u'_{x'x'} + \frac{U}{L^2}u'_{y'y'}\right)$$

   We can now drop the primes and simplify things to get,

$$u_t + uu_x + vu_y = -\frac{[p]}{\rho U^2}p_x + \frac{1}{\mathrm{Re}}\left(u_{xx} + u_{yy}\right)$$

   The other equations are done similarly and the final result in vector form is simply,

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u}\cdot\nabla)\mathbf{u} = -\frac{[p]}{\rho U^2}\nabla p + \frac{1}{\mathrm{Re}}\nabla^2\mathbf{u}$$

   (a) If Re is large, then we'll want to balance the inertial (left) terms with the pressure term. Thus $[p] = \rho U^2$, and

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u}\cdot\nabla)\mathbf{u} = -\nabla p + \frac{1}{\mathrm{Re}}\nabla^2\mathbf{u}$$

   The leading-order behaviour should correspond to inviscid flow (except near the boundaries of the object where viscosity becomes important).

   (b) If Re is small, then we'll want to balance the pressure term with the viscous terms. Thus $[p] = \rho U^2/\mathrm{Re} = \mu U/L$ and,

$$\mathrm{Re}\left(\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u}\cdot\nabla)\mathbf{u}\right) = -\nabla p + \nabla^2\mathbf{u}$$

   and the leading-order behaviour should correspond to the *slow-flow* equations,

$$\nabla \cdot \mathbf{u} = 0$$
$$\nabla^2 \mathbf{u} = \nabla p$$

3. PS3, Q2: **Boundary Layer on a Plate**

Consider two-dimensional steady viscous flow of a uniform stream with velocity $U\mathbf{i}$ past a semi-infinite plate at $y = 0$, $x > 0$.

(a) Begin with the dimensional 2D Navier-Stokes equations. By scaling $x,\ y \sim L$, $u,\ v \sim U$, and choosing an appropriate scalings for p, derive the non-dimensional equation,

$$(\mathbf{u} \cdot \nabla)\mathbf{u} = -\nabla p + \epsilon \nabla^2 \mathbf{u}$$

where $1/\epsilon = \text{Re} \equiv UL/\nu \gg 1$. What is the inviscid solution?

(b) By re-scaling $y = \delta Y$ and $v = \delta V$, and choosing δ appropriately, derive Prandtl's boundary layer equations valid near the plate,

$$u\frac{\partial u}{\partial x} + V\frac{\partial u}{\partial Y} = -\frac{\partial p}{\partial x} + \frac{\partial^2 u}{\partial Y^2}$$
$$0 = -\frac{\partial p}{\partial Y}$$
$$\frac{\partial u}{\partial x} + \frac{\partial V}{\partial Y} = 0$$

Now let $u = \Psi_Y$ and $V = -\Psi_x$ to get,

$$\Psi_Y \Psi_{xY} - \Psi_x \Psi_{YY} = -p_x + \Psi_{YYY}$$

Clearly state the boundary and matching conditions. Argue that $p_x = 0$

(c) Verify that there is a similarity solution of the form $\Psi(x, Y) = x^{1/2} f(\eta)$, $Y = x^{1/2}\eta$ that leads to the Blasius equation,

$$f''' + \frac{1}{2} f f'' = 0$$

Clearly state the boundary and matching conditions on f.

(d) Show that the dimensional drag per unit length on the plate is,

$$-F_1 = -\frac{U^2 \rho}{\sqrt{\text{Re}}} \frac{f''(0)}{x^{1/2}}$$

Solution:

(a) You did the nondimensionalisation on last week's worksheet. The inviscid solution is simply $\mathbf{u} = [1, 0, 0]$

(b) Let's do the x-momentum equation as an example. Under the re-scaling $y = \delta Y$ and $v = \delta V$, we get

$$u\frac{\partial u}{\partial x} + V\frac{\partial u}{\partial Y} = -\frac{\partial p}{\partial x} + \epsilon\left(\frac{\partial^2 u}{\partial x^2} + \frac{1}{\delta^2}\frac{\partial^2 u}{\partial Y^2}\right)$$

Now clearly, for δ small, the second viscous term dominates the first (remember: derivatives in y make things *larger*). So we should choose $\delta = \epsilon$, getting

$$u\frac{\partial u}{\partial x} + V\frac{\partial u}{\partial Y} = -\frac{\partial p}{\partial x} + \frac{\partial^2 u}{\partial Y^2}$$

The rest of the Navier-Stokes equations follow easily. The boundary condition on the plate $Y = 0$ is that $u = v = 0$. This is equivalent to requiring that the streamfunction is *constant* on the plate, which we can choose to be $\Psi = 0$. To match as $Y \to \infty$, we then need $\Psi_Y \to 1$ or $\Psi \to Y$.

Finally, taking the streamfunction equation to $Y \to \infty$ tells us all the terms involving derivatives of Ψ disappear, leaving $p_x \to 0$ as $Y \to \infty$. Since the limit is independent of x this implies $p_x = 0$ everywhere.

(c) The substitution is straightforward but messy (be sure to keep things organised, lest you make mistakes!). The boundary conditions require,

$$f(0) = 0 \text{ and } f(\infty) = \eta$$

(d) The dimensional drag is,

$$-F_1 = -\mu\frac{U}{L}\left(\frac{1}{\sqrt{\text{Re}}}\frac{\partial u}{\partial Y} + \sqrt{\text{Re}}\frac{\partial v}{\partial x}\right)$$

The second term is zero on the plate (why?). So we have

$$-F_1 = -\frac{U^2\rho}{\sqrt{\text{Re}}}\frac{f''(0)}{x^{1/2}}$$

after writing $\mu = UL\rho/\text{Re}$

4. PS4, Q1: **Slow-Flow Equations**

First, consider three-dimensional inviscid and incompressible flow of a uniform stream with velocity U past a sphere/cylinder with radius a located at the origin. Recall that the leading-order slow flow equations:

$$\nabla \cdot \mathbf{u} = 0, \qquad \nabla^2\mathbf{u} = \nabla p$$

Now put the slow-flow equations in the alternative form

$$\nabla \cdot \mathbf{u} = 0 \qquad \text{curl}^3\mathbf{u} = 0. \tag{1}$$

You may use the vector identity: $\nabla^2\mathbf{u} = \nabla(\nabla \cdot \mathbf{u}) - \nabla \times (\nabla \times \mathbf{u})$

Solution: We simply take the curl of the second equation, noting that $\nabla \times \nabla p = 0$ and,

$$\nabla \times \nabla^2 \mathbf{u} = \nabla \times \Big[\nabla(\nabla \cdot \mathbf{u}) - \nabla \times (\nabla \times \mathbf{u})\Big] = -\text{curl}^3 \mathbf{u} = 0$$

5. PS4, Q1: **[Low-Re Flow Past Circular Cylinder]**

Now consider *two-dimensional* flow of a uniform stream with velocity $U\mathbf{i}$ past a circular cylinder of radius a located at the origin in the polar coordinate system (r, θ).

(a) Begin with the non-dimensional slow-flow equations you derived earlier and show that if we have a streamfunction

$$\mathbf{u} = [u, v] = \left[\frac{\partial \psi}{\partial y}, -\frac{\partial \psi}{\partial x}\right],$$

this leads to the biharmonic equation

$$\nabla^4 \psi = 0.$$

Now write down *two* boundary conditions for ψ on $r = 1$ and the far-field condition for ψ as $r \to \infty$.

(b) By separating variables $\psi = f(r) \sin\theta$ and letting $f(r) = r^n$, show that

$$f = \frac{A}{r} + Br + Cr\log r + Dr^3.$$

You will find it useful to write the Laplacian in the form

$$\nabla^2 = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}$$

(c) Demonstrate the Stokes/Oseen paradox and briefly explain how it can be resolved.

Solution:

(a) You can do this question directly by substituting in the expression for $\mathbf{u}$ in terms of the streamfunction.

$$\omega = \nabla \times \mathbf{u} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial \psi}{\partial y} & -\frac{\partial \psi}{\partial y} & 0 \end{vmatrix} = [0, 0, -\nabla^2 \psi] \tag{2}$$

Applying the curl twice more we get

$$\nabla \times \nabla \times (-\nabla^2 \psi) = [0, 0, \psi_{xxxx} + 2\psi_{xxyy} + \psi_{yyyy}] = [0, 0, \nabla^2 \cdot \nabla^2 \psi],$$

or $\nabla^4 \psi = 0$ as desired.

The boundary conditions on $r = 1$ is $\psi_r = 0$ (no flux) and ψ_θ (no slip). Together, they imply that (without loss of generality) $\psi = 0$, and we will keep the $\psi_r = 0$ as well. As $r \to \infty$, we'd expect $\mathbf{u} = \mathbf{i}$ so $\psi_y = 1$ so $\psi = y + \text{constant} \sim r \sin\theta$ as $r \to \infty$. Thus

$$\psi = \frac{\partial \psi}{\partial r} = 0 \text{ on } r = 1 \text{ and } \psi \to r\sin\theta \text{ as } r \to \infty$$

(b) First note that

$$\nabla^2\psi = \left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right] f(r)\sin\theta = \left[\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} - \frac{1}{r^2}\right] f(r)\sin\theta,$$

so in fact,

$$\nabla^4\psi = \left[\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} - \frac{1}{r^2}\right]^2 f(r)\sin\theta = 0 \Rightarrow \left[\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} - \frac{1}{r^2}\right]^2 f(r) = 0$$

Now using $f(r) = r^n$ we get

$$\left[\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} - \frac{1}{r^2}\right]\left[(n-1)(n+1)r^{n-2}\right] = (n+1)(n-1)^2(n-3)r^{n-4} = 0$$

So we get $r = -1, 1, 1, 3$. Therefore,

$$f(r) = \frac{A}{r} + Br + Cr\log r + Dr^3$$

since the double roots at $n = 1$ means we should multiply the r term by a $\log r$.

The condition that $f(\infty) \sim r$ (or that $f'(\infty) = 1$) means $C = D = 0$ and $B = 1$. However, applying the condition that $f(1) = f'(1) = 0$ implies $A = B = 0$. So there is no nontrivial solution.

Stokes' Paradox

Stokes' paradox arises because in the slow-flow problem we have

$$\frac{\epsilon}{r}\frac{\partial(\psi, \nabla^2\psi)}{\partial(\theta, r)} = \nabla^2\psi.$$

where $\epsilon = \text{Re}$. Usually, we assume the left-hand side is negligible. Unfortunately, if $r \sim O(1/\epsilon)$, then the left side balances the right (you have to expand everything out to see this, since there are rs in the operators!)

Thus as $r \to \infty$ the slow solution doesn't quite match directly with the solution at $r = \infty$. There is a boundary layer at infinity. We will need let $r = \overline{r}/\epsilon$, and solve for the solution to this now-balanced (Oseen) equation with the condition that $\psi \sim \overline{r}\sin\theta/\epsilon$ as $\overline{r} \to \infty$. The matching between *this* asymptotic solution and the slow-flow solution as $\overline{r} \to 0$ can now be done.