1. PS1: **Deriving Navier Stokes**

   Derive the Navier-Stokes equations (continuity and momentum) for an incompressible, Newtonian, and viscous fluid with body force $\mathbf{g}$. Write down what it means for the fluid to be incompressible in terms of the density, ρ. You can use the standard lemmas (*e.g.* Reynold's Transport Theorem, material derivative, etc.), but clearly quote anything you use.

2. PS2, Q6: **Practise non-dimensionlising equations**

   Start from the **dimensional** Navier-Stokes equations you derived in the last question. Consider **two-dimensional** flow past an obstacle of typical size L, with speed U far away. Derive the non-dimensional equations in terms of the Reynolds number, $\text{Re} = UL/\nu$ for,

   (a) $\text{Re} \gg 1$ leading to inviscid flow

   (b) $\text{Re} \ll 1$ leading to slow flow

   For each above case, you will have to select the correct choice of typical pressure p. Also for each case, explain what the *leading-order* behaviour should be.

3. PS3, Q2: **Boundary Layer on a Plate**

   Consider two-dimensional steady viscous flow of a uniform stream with velocity $U\mathbf{i}$ past a semi-infinite plate at $y = 0$, $x > 0$.

   (a) Begin with the dimensional 2D Navier-Stokes equations. By scaling $x,\ y \sim L$, $u,\ v \sim U$, and choosing an appropriate scalings for p, derive the non-dimensional equation,

   $$(\mathbf{u} \cdot \nabla)\mathbf{u} = -\nabla p + \epsilon \nabla^2 \mathbf{u}$$

   where $1/\epsilon = \text{Re} \equiv UL/\nu \gg 1$. What is the inviscid solution?

   (b) By re-scaling $y = \delta Y$ and $v = \delta V$, and choosing δ appropriately, derive Prandtl's boundary layer equations valid near the plate,

   $$u\frac{\partial u}{\partial x} + V\frac{\partial u}{\partial Y} = -\frac{\partial p}{\partial x} + \frac{\partial^2 u}{\partial Y^2}$$
   $$0 = -\frac{\partial p}{\partial Y}$$
   $$\frac{\partial u}{\partial x} + \frac{\partial V}{\partial Y} = 0$$

   Now let $u = \Psi_Y$ and $V = -\Psi_x$ to get,

   $$\Psi_Y \Psi_{xY} - \Psi_x \Psi_{YY} = -p_x + \Psi_{YYY}$$

   Clearly state the boundary and matching conditions. Argue that $p_x = 0$

   (c) Verify that there is a similarity solution of the form $\Psi(x, Y) = x^{1/2} f(\eta)$, $Y = x^{1/2}\eta$ that leads to the Blasius equation,

   $$f''' + \frac{1}{2} f f'' = 0$$

   Clearly state the boundary and matching conditions on f.

   (d) Show that the dimensional drag per unit length on the plate is,

   $$-F_1 = -\frac{U^2 \rho}{\sqrt{\text{Re}}} \frac{f''(0)}{x^{1/2}}$$

4. PS4, Q1: **Slow-Flow Equations**

   First, consider three-dimensional inviscid and incompressible flow of a uniform stream with velocity U past a sphere/cylinder with radius a located at the origin. Recall that the leading-order slow flow equations:

$$\nabla \cdot \mathbf{u} = 0, \qquad \nabla^2 \mathbf{u} = \nabla p$$

   Now put the slow-flow equations in the alternative form

$$\nabla \cdot \mathbf{u} = 0 \qquad \mathrm{curl}^3 \mathbf{u} = 0. \tag{1}$$

   You may use the vector identity: $\nabla^2 \mathbf{u} = \nabla(\nabla \cdot \mathbf{u}) - \nabla \times (\nabla \times \mathbf{u})$

5. PS4, Q1: **[Low-Re Flow Past Circular Cylinder]**

   Now consider *two-dimensional* flow of a uniform stream with velocity $U\mathbf{i}$ past a circular cylinder of radius a located at the origin in the polar coordinate system (r, θ).

   (a) Begin with the non-dimensional slow-flow equations you derived earlier and show that if we have a streamfunction

$$\mathbf{u} = [u, v] = \left[\frac{\partial \psi}{\partial y}, -\frac{\partial \psi}{\partial x}\right],$$

   this leads to the biharmonic equation

$$\nabla^4 \psi = 0.$$

   Now write down *two* boundary conditions for ψ on $r = 1$ and the far-field condition for ψ as $r \to \infty$.

   (b) By separating variables $\psi = f(r) \sin\theta$ and letting $f(r) = r^n$, show that

$$f = \frac{A}{r} + Br + Cr \log r + Dr^3.$$

   You will find it useful to write the Laplacian in the form

$$\nabla^2 = \frac{1}{r}\frac{\partial}{\partial r}\left(r \frac{\partial}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}$$

   (c) Demonstrate the Stokes/Oseen paradox and briefly explain how it can be resolved.