B6a Viscous Flows

1. **Governing Equations**: Write down the formulae for each of the following
   (a) The **Material Derivative** for a function $f(\mathbf{x}, t)$ and velocity $\mathbf{u}(\mathbf{x}, t)$.
   (b) **Reynold's Transport Theorem** a function $f(\mathbf{x}, t)$, velocity $\mathbf{u}(\mathbf{x}, t)$, and volume, $V(t)$
   (c) The **Continuity Equation** for compressible flow with density ρ and velocity $\mathbf{u}$
   (d) The **Continuity Equation** for incompressible flow
   (e) The **Navier-Stokes Equations** for incompressible flow with viscosity μ and gravity $\mathbf{g}$.

Solution:

Material Derivative

$$\frac{Df}{Dt} = \frac{\partial f}{\partial t} + (\mathbf{u} \cdot \nabla) f$$

Reynold's Transport Theorem

$$\frac{d}{dt} \iiint_{V(t)} f \, dV = \iiint_{V(t)} \frac{\partial f}{\partial t} + \nabla \cdot (f\mathbf{u}) \, dV$$

Continuity Equation (compressible)

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0$$

Continuity Equation (incompressible)

$$\nabla \cdot \mathbf{u} = 0$$

Navier-Stokes Equations (incompressible)

$$\rho \left(\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla) \mathbf{u} \right) = -\nabla p + \rho \mathbf{g} + \mu \nabla^2 \mathbf{u}$$

2. **Continuity equation** Derive the continuity equation for a compressible fluid with velocity $\mathbf{u}(\mathbf{x}, t)$ and density $\rho(\mathbf{x}, t)$. Afterwards, derive the incompressible version of the continuity equation. Note that you should also derive Reynold's Transport Theorem.

Solution: By mass conservation and the Transport Theorem,

$$\frac{d}{dt} \iiint_{V(t)} \rho \, dV = \iiint_{V(t)} \frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) \, dV = 0,$$

thus we have the Continuity Equation,

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0.$$

We can write the left-hand side in the alternative form,

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = \frac{\partial \rho}{\partial t} + (\mathbf{u} \cdot \nabla)\rho + \rho(\nabla \cdot \mathbf{u}) = \frac{D\rho}{dt} + \rho(\nabla \cdot \mathbf{u}) = 0$$

Therefore, if the flow is incompressible and $D\rho/Dt = 0$, then

$$\nabla \cdot \mathbf{u} = 0$$

3. **Navier-Stokes equations** Beginning with Newton's second law for a material volume $V(t)$, derive the Navier-Stokes equations for an incompressible *Newtonian* fluid. You may find two identities helpful:

$$\begin{aligned} \nabla(fg) &= f(\nabla g) + g(\nabla f) \\ \nabla \cdot (f\mathbf{G}) &= f(\nabla \cdot \mathbf{G}) + \mathbf{G} \cdot (\nabla f) \end{aligned}$$

Solution:

$$\begin{aligned} \text{Change in momentum} &= \text{Sum of forces in } i^{th} \text{ direction} \\ \underbrace{\frac{d}{dt}\iiint_{V(t)} \rho u_i \, dV}_{\text{momentum}} &= \underbrace{\iiint_{V(t)} \rho g_i \, dV}_{\text{body forces}} + \underbrace{\iint_{\partial V(t)} F_i \, dS}_{\text{surface forces}} \\ \underbrace{\iiint_{V(t)} \frac{\partial(\rho u_i)}{\partial t} + \nabla \cdot (\rho u_i \mathbf{u}) \, dV}_{①} &= \iiint_{V(t)} \rho g_i \, dV + \underbrace{\iint_{\partial V(t)} \sigma_{ij} n_j \, dS}_{②} \end{aligned}$$

where we used the Transport Theorem in the second line. Then we can expand,

$$\nabla \cdot \Big((\rho u_i)\mathbf{u} \Big) = \mathbf{u} \cdot \nabla(\rho u_i) + (\rho u_i)(\nabla \cdot \mathbf{u}) = \mathbf{u} \cdot (u_i \nabla \rho + \rho \nabla u_i) = u_i(\mathbf{u} \cdot \nabla)\rho + \rho(\mathbf{u} \cdot \nabla)u_i$$

Using $D\rho/Dt = 0$, we can write

$$\begin{aligned} ① &= \iiint_{V(t)} u_i \left(\frac{\partial \rho}{\partial t} + (\mathbf{u} \cdot \nabla)\rho \right) + \rho \left(\frac{\partial u_i}{\partial t} + (\mathbf{u} \cdot \nabla)u_i \right) \\ &= \iiint_{V(t)} \rho \frac{Du_i}{Dt} \, dV \end{aligned}$$

Whereas by the Divergence Theorem,

$$\begin{aligned} ② = \iiint_{V(t)} \frac{\partial \sigma_{ij}}{\partial x_j} \, dV &= \iiint_{V(t)} \frac{\partial}{\partial x_j} \left(-p\delta_{ij} + \mu \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) \right) \, dV \\ &= \iiint_{V(t)} -\frac{\partial p}{\partial x_i} + \mu \frac{\partial^2 u_i}{\partial x_j \partial x_j} + \mu \frac{\partial^2 u_j}{\partial x_j \partial x_i} \, dV \\ &= \iiint_{V(t)} -\frac{\partial p}{\partial x_i} + \mu \nabla^2 u_i + \mu \frac{\partial}{\partial x_i}(\nabla \cdot \mathbf{u}) \, dV \\ &= \iiint_{V(t)} -\frac{\partial p}{\partial x_i} + \mu \nabla^2 u_i \end{aligned}$$

Putting it all together, equating the integrands, and returning to vector form, we have

$$\rho \left(\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u} \right) = -\nabla p + \rho \mathbf{g} + \mu \nabla^2 \mathbf{u}.$$

4. **Rayleigh flow and similarity solutions**

Consider incompressible Newtonian fluid at rest in the region $y > 0$ above a rigid plate at $y = 0$. At time $t = 0$ the plate is jerked into motion in the x-direction with constant velocity U. There are no external body forces or applied pressure gradients.

(a) By assuming the flow is unidirectional $\mathbf{u} = [u(y,t), 0]$, and using the **dimensional** Navier-Stokes equations, derive the (one) equation that governs the fluid. What are the (2) boundary and (1) initial conditions?

(b) Assume that $u(y,t) = Uf(\eta)$ where $\eta = y/\delta(t)$. By substituting this form into the equations, find a choice of $\delta(t)$ that will allow you to construct a similarity solution and hence the single ordinary differential equation for $f(\eta)$, given by

$$f'' + 2\eta f' = 0.$$

(c) Solve for $f(\eta)$

Solution:

(a)

$$\frac{\partial u}{\partial t} = \nu \frac{\partial^2 u}{\partial y^2},$$

with $u(0,t) = U$ and $u(\infty, t) = 0$ for $t > 0$ and $u(y, 0) = 0$

(b)

$$\begin{aligned}
\frac{\partial u}{\partial t} &= \frac{\partial \eta}{\partial t}\frac{\partial (Uf)}{\partial \eta} = -\frac{Uy}{\delta^2}\delta'(t)f'(\eta) = -U\eta\frac{\delta'}{\delta}f'(\eta) \\
\frac{\partial u}{\partial y} &= \frac{\partial \eta}{\partial y}\frac{\partial (Uf)}{\partial y} = \frac{U}{\delta}f'(\eta) \\
\frac{\partial^2 u}{\partial y^2} &= \frac{U}{\delta^2}f''(\eta)
\end{aligned}$$

Thus the differential equation becomes,

$$-\eta\frac{\delta'}{\delta}f' = \frac{\nu}{\delta^2}f'' \Rightarrow -\eta\left(\frac{\delta\delta'}{\nu}\right)f' = f''$$

The right-hand side is independent of t and ν, and we want the same for the left. Thus, we must have the expression in the brackets equal to a constant, say

$$\delta\delta' = \frac{\nu}{8} \Rightarrow \delta = \sqrt{4\nu t}$$

With this choice, the equation now becomes,

$$f'' + 2\eta f' = 0,$$

with $f(0) = 1$ and $f(\infty) = 0$.

(c) By integrating factors,

$$f(\eta) = C \int_{\eta}^{\infty} e^{-s^2}\, ds + D$$

for constants C and D. By the condition $f(\infty) = 0$, D is zero. Also, remembering that,

$$\int_{-\infty}^{\infty} e^{-s^2}\, ds = \sqrt{\pi}$$

and $f(0) = 1$, then $C = 2/\sqrt{\pi}$.

5. **Stokes layer**

Incompressible Newtonian fluid occupies the region $y > 0$ above a rigid plate at $y = 0$ which oscillates to and fro in the x direction with velocity $U \cos \Omega t$. There are no external body forces and there is no applied pressure gradient. The flow is unidirectional with velocity $\mathbf{u} = u(y,t)\mathbf{i}$.

(a) By assuming uni-directionality, show that the velocity must satisfy the diffusion equation

$$\frac{\partial u}{\partial t} = \nu \frac{\partial^2 u}{\partial y^2}.$$

with $u(0,t) = U \cos \Omega t$ and $u(\infty, t) = 0$ for $-\infty < t < \infty$.

(b) By seeking a solution of the from $u = \Re(Uf(y)e^{i\Omega t})$, show that

$$u = Ue^{-ky} \cos(ky - \Omega T),$$

where $k = \sqrt{\Omega/2\nu}$.

(c) Determine the vorticity $\omega = \nabla \wedge \mathbf{u}$ and show that its magnitude is exponentially small except in a layer near the boundary. What is the size of this layer? Sketch the velocity profile at time $t = 0$, indicating the Stokes layer in which the vorticity is significant.

Solution: Inquire.

6. (Optional) **Stress tensor & forces**

   (a) State the form of the stress tensor, σ_{ij} for an incompressible Newtonian fluid. For this stress tensor, what is the force $\mathbf{F}$, felt by a surface with outward normal $\mathbf{n}$?

   (b) Consider two-dimensional flow between two plates at $y = -1$ and $y = 1$ with velocity $\mathbf{u}(x, y) = [1 - y^2, 0]$ and constant pressure $p = 0$. Find the force per unit area at the point $(1, 1/2)$ on a surface element whose outward normal points $30°$ to the direction of the flow.

Solution:

(a) The stress tensor for an incompressible Newtonian fluid is,

$$\begin{aligned}\sigma_{ij} &= \text{stress in } i^{th} \text{ direction on surface with } j^{th} \text{ normal} \\ &= \underbrace{-p\delta_{ij}}_{\text{expansion/contraction}} \quad + \mu \underbrace{\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)}_{\text{shears/rotations}}\end{aligned}$$

The force in the i^{th} direction is found by summing over all the surfaces with normals n_j,

$$F_i = \sigma_{ij} n_j = \sigma_{i1} n_1 + \sigma_{i3} n_1 + \sigma_{i3} n_3$$

(b) Let's first write out the entire tensor,

$$\begin{aligned}\sigma &= \begin{bmatrix} -p + 2\mu u_x & \mu\left(u_y + v_x\right) \\ \mu\left(u_y + v_x\right) & -p + 2\mu v_y \end{bmatrix} \\ &= \begin{bmatrix} 0 & -2\mu y \\ -2\mu y & 0 \end{bmatrix}\end{aligned}$$

Now the (unit) outward normal of the surface element is

$$\mathbf{n} = \left[\cos\left(\frac{\pi}{6}\right), \sin\left(\frac{\pi}{6}\right)\right] = \left[\frac{\sqrt{3}}{2}, \frac{1}{2}\right]$$

Substituting $y = 1$ and multiplying σ with $\mathbf{n}$ gives,

$$\mathbf{F} = \frac{1}{2}\begin{bmatrix} 0 & -\mu \\ -\mu & 0 \end{bmatrix}\begin{bmatrix} \sqrt{3} \\ 1 \end{bmatrix} = -\frac{\mu}{2}\begin{bmatrix} 1 \\ \sqrt{3} \end{bmatrix}$$

7. (Optional) **Energy equation**

Derive the conservation of energy equation for a material volume $V(t)$ of an incompressible conducting fluid in the form,

$$\rho c_v \frac{DT}{Dt} = k\nabla^2 T + \Phi,$$

where

$$\Phi = \frac{\mu}{2} \sum_{i,j=1}^{3} \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right)^2,$$

by making the following assumptions:

1. There is an external body force **F**
2. There are no external energy sources
3. The fluid is incompressible
4. The specific heat c_v and thermal conductivity k is constant

Solution: Conservation of energy tells us that,

Change in energy of $V(t)$ = Heat flux into $V(t)$ + Work done on $V(t)$

We can write this as

$$\underbrace{\frac{d}{dt} \iiint\limits_{V(t)} \left[\rho c_v T + \frac{1}{2}\rho u_i^2 \right] dV}_{①} = \underbrace{\iint\limits_{\partial V(t)} k \frac{\partial T}{\partial x_j} n_j \, dS}_{②} + \underbrace{\iiint\limits_{V(t)} \rho u_i F_i \, dV}_{③} + \underbrace{\iint\limits_{\partial V(t)} u_i \sigma_{ij} n_j \, dS}_{④},$$

where T is the temperature, c_v is the specific heat, k is the thermal conductivity and,

① is the rate of change of energy (heat and kinetic)

② is the rate at which heat is conducted into $V(t)$ across the boundary, since $\mathbf{q} \cdot (-\mathbf{n}) = -k\nabla T \cdot (-\mathbf{n})$ by Fourier's Law.

③ is the rate at which work is done by external body forces

④ is the rate at which work is done by surface forces

Note that with incompressibility and Reynold's Transport Theorem,

$$\frac{d}{dt}\iiint\limits_{V(t)} \rho F \, dV = \iiint\limits_{V(t)} \rho \frac{DF}{Dt} \, dV,$$

so we may now write the left-hand-side as,

$$① = \iiint\limits_{V(t)} \rho \frac{D}{Dt}\left[c_v T + \frac{1}{2}u_i^2\right] \, dV.$$

Applying the divergence theorem to ② and ④ then puts the right-hand-side in the form,

$$\iiint\limits_{V(t)} \left[\frac{\partial}{\partial x_j}\left(k\frac{\partial T}{\partial x_j} + u_i\sigma_{ij}\right) - \rho u_i F_i\right] \, dV$$

Now equate the integrands and a few product rules yields

$$\rho c_v \frac{DT}{Dt} + u_i \underbrace{\left(\rho\frac{Du_i}{Dt} - \frac{\partial \sigma_{ij}}{\partial x_j} - \rho F_i\right)}_{=0 \text{ by momentum equation}} = k\nabla^2 T + \sigma_{ij}\frac{\partial u_i}{\partial x_j}.$$

This leaves us with,

$$\rho c_v \frac{DT}{Dt} = k\nabla^2 T + \Phi, \quad \text{where } \Phi = \sigma_{ij}\frac{\partial u_i}{\partial x_j}.$$

Finally, for an incompressible Newtonian fluid, we can write,

$$\sigma_{ij} = -p\delta_{ij} + 2\mu e_{ij}, \quad \text{where } e_{ij} = \frac{\mu}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_i}{\partial x_j}\right)$$

and manipulate the dissipation as follows,

$$\begin{aligned}
\Phi &= \underbrace{-p\delta_{ij}\frac{\partial u_i}{\partial x_j}}_{\text{zero since } \nabla\cdot\mathbf{u}=0} + \underbrace{2\mu e_{ij}\frac{\partial u_i}{\partial x_j}}_{\text{split me and swap dummy indices}} \\
&= \underbrace{\mu e_{ij}\frac{\partial u_i}{\partial x_j} + \mu e_{ji}\frac{\partial u_j}{\partial x_i}}_{\text{note } e_{ij}=e_{ji}} \\
&= \mu e_{ij}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) = 2\mu e_{ij}e_{ij} = \frac{\mu}{2}\sum_{i,j=1}^{3}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)^2
\end{aligned}$$