B6a Viscous Flows

1. **Governing Equations**: Write down the formulae for each of the following
   (a) The **Material Derivative** for a function $f(\mathbf{x}, t)$ and velocity $\mathbf{u}(\mathbf{x}, t)$.
   (b) **Reynold's Transport Theorem** a function $f(\mathbf{x}, t)$, velocity $\mathbf{u}(\mathbf{x}, t)$, and volume, $V(t)$
   (c) The **Continuity Equation** for compressible flow with density ρ and velocity $\mathbf{u}$
   (d) The **Continuity Equation** for incompressible flow
   (e) The **Navier-Stokes Equations** for incompressible flow with viscosity μ and gravity $\mathbf{g}$.

2. **Continuity equation** Derive the continuity equation for a compressible fluid with velocity $\mathbf{u}(\mathbf{x}, t)$ and density $\rho(\mathbf{x}, t)$. Afterwards, derive the incompressible version of the continuity equation. Note that you should also derive Reynold's Transport Theorem.

3. **Navier-Stokes equations** Beginning with Newton's second law for a material volume $V(t)$, derive the Navier-Stokes equations for an incompressible *Newtonian* fluid. You may find two identities helpful:

$$\begin{aligned}\nabla(fg) &= f(\nabla g) + g(\nabla f) \\ \nabla \cdot (f\mathbf{G}) &= f(\nabla \cdot \mathbf{G}) + \mathbf{G} \cdot (\nabla f)\end{aligned}$$

4. **Rayleigh flow and similarity solutions**

Consider incompressible Newtonian fluid at rest in the region $y > 0$ above a rigid plate at $y = 0$. At time $t = 0$ the plate is jerked into motion in the x-direction with constant velocity U. There are no external body forces or applied pressure gradients.

(a) By assuming the flow is unidirectional $\mathbf{u} = [u(y,t), 0]$, and using the **dimensional** Navier-Stokes equations, derive the (one) equation that governs the fluid. What are the (2) boundary and (1) initial conditions?

(b) Assume that $u(y,t) = Uf(\eta)$ where $\eta = y/\delta(t)$. By substituting this form into the equations, find a choice of $\delta(t)$ that will allow you to construct a similarity solution and hence the single ordinary differential equation for $f(\eta)$, given by

$$f'' + 2\eta f' = 0.$$

(c) Solve for $f(\eta)$

5. **Stokes layer**

Incompressible Newtonian fluid occupies the region $y > 0$ above a rigid plate at $y = 0$ which oscillates to and fro in the x direction with velocity $U \cos \Omega t$. There are no external body forces and there is no applied pressure gradient. The flow is unidirectional with velocity $\mathbf{u} = u(y,t)\mathbf{i}$.

(a) By assuming uni-directionality, show that the velocity must satisfy the diffusion equation

$$\frac{\partial u}{\partial t} = \nu \frac{\partial^2 u}{\partial y^2}.$$

with $u(0,t) = U \cos \Omega t$ and $u(\infty, t) = 0$ for $-\infty < t < \infty$.

(b) By seeking a solution of the from $u = \Re(Uf(y)e^{\Omega t})$, show that

$$u = Ue^{-ky} \cos(ky - \Omega T),$$

where $k = \sqrt{\Omega/2\nu}$.

(c) Determine the vorticity $\omega = \nabla \wedge \mathbf{u}$ and show that its magnitude is exponentially small except in a layer near the boundary. What is the size of this layer? Sketch the velocity profile at time $t = 0$, indicating the Stokes layer in which the vorticity is significant.

6. (Optional) **Stress tensor & forces**

   (a) State the form of the stress tensor, σ_{ij} for an incompressible Newtonian fluid. For this stress tensor, what is the force $\mathbf{F}$, felt by a surface with outward normal $\mathbf{n}$?

   (b) Consider two-dimensional flow between two plates at $y = -1$ and $y = 1$ with velocity $\mathbf{u}(x, y) = [1 - y^2, 0]$ and constant pressure $p = 0$. Find the force per unit area at the point $(1, 1/2)$ on a surface element whose outward normal points 30° to the direction of the flow.

7. (Optional) **Energy equation**

Derive the conservation of energy equation for a material volume $V(t)$ of an incompressible conducting fluid in the form,

$$\rho c_v \frac{DT}{Dt} = k\nabla^2 T + \Phi,$$

where

$$\Phi = \frac{\mu}{2} \sum_{i,j=1}^{3} \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right)^2,$$

by making the following assumptions:

1. There is an external body force $\mathbf{F}$
2. There are no external energy sources
3. The fluid is incompressible
4. The specific heat c_v and thermal conductivity k is constant