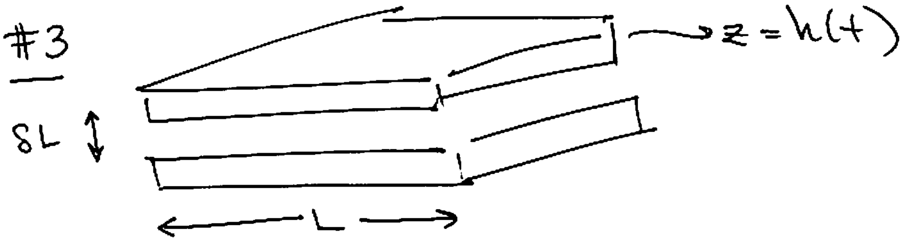
#3




Scale $u, v \sim U \quad x, y \sim L, \quad z \sim \delta L. \quad t \sim \frac{L}{U}$

Note that by cty: $u_x + v_y + w_z = 0$

$\Rightarrow$ need $w \sim \delta U$

Scale $\frac{dz}{dt} = h'(t) \sim \delta U$ (note this is given by condition $w = h(t)$ on $z = h(t)$)

Dimensional NS:

$$\frac{\partial u}{\partial t} + u u_x + v u_y + w u_z = -\frac{1}{\rho} p_x + \mu (u_{xx} + u_{yy} + u_{zz})$$

$$\Rightarrow \frac{U^2}{L}\left\{ u_t + u u_x + v u_y + w u_z \right\} = -\frac{1}{\rho} \frac{[p]}{L} p_x + \mu U \left( \frac{1}{L^2} u_{xx} + \frac{1}{L^2} u_{yy} + \frac{1}{\delta^2 L^2} u_{zz} \right)$$

$$\frac{\delta^2 U L \rho}{\mu}\left\{ u_t + u u_x + v u_y + w u_z \right\} = -\frac{\delta^2 L}{\mu U} [p]\, p_x + \Big( u_{zz} + \underbrace{\cdots}_{O(\delta^2)} \Big)$$

choose $[p] = \frac{\mu U}{\delta^2 L}$. Then if $\delta^2 \frac{U L \rho}{\mu} \ll 1$

$$\Rightarrow \boxed{0 = -p_x + u_{zz}}$$

By symmetry: $\boxed{0 = -p_y + v_{zz}}$

In the z-dir.:

$$\underbrace{w_t + u w_x + v w_y + w w_z}_{O(\delta)} = -\frac{1}{\rho} \underbrace{p_z}_{O(\frac{1}{\delta^3}) \Rightarrow \text{dominates}} + \mu \Big( \underbrace{w_{xx} + w_{yy}}_{O(\delta)} + \underbrace{w_{zz}}_{O(\frac{1}{\delta})} \Big)$$

$\therefore$ $\boxed{p_z = 0}$

<u>B.C.s</u>: $u = v = w = 0$ on $z = 0$.

$u = v = 0$, $w = \frac{dh}{dt}$ on $z = h(t)$

For next bit, int. cont. eqn:

$$0 = \int_0^{h(t)} u_x + v_y + w_z \cdot dz = \frac{\partial}{\partial x}\int_0^h u \cdot dz - \cancel{u \cdot \frac{dh}{dt}}\Big|_{z=h}$$

$$+ \frac{\partial}{\partial y}\int_0^h v\, dz - \cancel{v \cdot \frac{dh}{dt}}\Big|_{z=h}$$

$$+ w\Big|_0^h$$

note $\frac{\partial}{\partial x}\int_0^{h(t)} f \cdot dz = \int_0^h \frac{\partial f}{\partial x} \cdot dz + \frac{\partial h}{\partial t} \cdot f\Big|_{h(t)}$

$$= \frac{\partial}{\partial x}(h\bar{u}) + \frac{\partial}{\partial y}(h\bar{v}) + \frac{dh}{dt} = 0.$$

$$\therefore \boxed{\frac{dh}{dt} + \nabla \cdot (h\bar{\underline{u}}) = 0}$$

For next bit, use:

$$u_{zz} = p_x \Rightarrow u = C \cdot p_x z(z-h)$$

$$u = \frac{1}{2\mu}\frac{\partial p}{\partial x}\, z(z-h)$$

$$\Rightarrow \bar{u} = \frac{1}{h}\int_0^h \frac{1}{2\mu} p_x\, \cancel{z}(z^2 - hz)\cdot dz$$

$$= \frac{p_x}{2\mu h}\left\{\frac{1}{3}h^3 - \frac{1}{2}h^3\right\} = \underline{\underline{\frac{-p_x \cdot h^2}{12\mu}}}$$

Similarly, $\bar{v} = \dfrac{-p_y h^2}{12\mu}$

$$\therefore\ \bar{\underline{u}} = \frac{-h^2}{12\mu}\nabla p.$$

b)

$p=0$, $\underline{u}\cdot\underline{e_r} = \dfrac{dR}{dt}$ on $r=R(t)$

$$\frac{dh}{dt} + \nabla\cdot\left(\frac{-h^3}{12\mu}\nabla p\right) = 0.$$

$$\Rightarrow \frac{dh}{dt} - \frac{h^3}{12\mu}\nabla^2 p = 0 \Rightarrow \nabla^2 p = \frac{12\mu}{h^3}\cdot\frac{dh}{dt}$$

since $\nabla^2 p = \dfrac{1}{r}\dfrac{\partial}{\partial r}\left(r\cdot\dfrac{\partial p}{\partial r}\right) = \dfrac{12\mu}{h^3}\cdot\dfrac{dh}{dt}$

$$\Rightarrow r\cdot\frac{\partial p}{\partial r} = \frac{6\mu}{h^3}\cdot\frac{dh}{dt} r^2 + B.$$

$$\frac{\partial p}{\partial r} = \frac{6\mu}{h^3}\frac{dh}{dt} r + \frac{B}{r} \Rightarrow p = \frac{3\mu}{h^3}\frac{dh}{dt} r^2 + B\log r + C$$

• $p = 0$ @ $r = R(t)$ & $p$ bounded $\Rightarrow B = 0$ &

$$p = \frac{3\mu}{h^3}\frac{dh}{dt}\left(r^2 - R^2\right)$$

Also, have kinematic condition

$$\bar{\underline{u}} \cdot e_r = \frac{dR}{dt}$$

$$\Rightarrow \quad -\frac{h^2}{12\mu}\nabla p \cdot e_r \quad \Rightarrow \quad -\frac{h^2}{12\mu}\cdot\frac{\partial p}{\partial r} = \frac{dR}{dt} \quad @ \quad r = R(t)$$

$$\Rightarrow \quad -\frac{h^2}{12\mu}\left[\frac{6\mu}{h^3}\cdot\frac{dh}{dt}\cdot R\right] = \frac{dR}{dt}$$

$$\Rightarrow \quad -\frac{1}{2h}\frac{dh}{dt}\cdot R = \frac{dR}{dt}$$

$$\Rightarrow \quad -\frac{1}{2h}\cdot\frac{dh}{dt} = \frac{1}{R}\frac{dR}{dt}$$

$$\Rightarrow \quad -\frac{1}{2}\underset{\text{fn of } t}{\log h} = \underset{\text{fn of } t}{\log R} + \cancel{\text{const}}.$$

$$\Rightarrow \quad \therefore \quad \log\left(R^2 \cdot h\right) = \text{const}$$

$$\Rightarrow \quad R^2 \cdot h = \text{const}$$

$$\Rightarrow R^2(t)h(t)\pi = \text{const} = \pi h(0) R^2(0)$$

global consv. of mass (Why?)