#4

BG à 2009

3. A thin layer of incompressible Newtonian fluid of constant density ρ and constant viscosity μ is sandwiched between two rigid parallel plates at $z = 0$ and $z = h(t)$. The plates are of typical lateral extent L and of typical separation δL, where $\delta = h(0)/L \ll 1$. The lower plate at $z = 0$ is stationary and the upper plate at $z = h(t)$ moves normally with typical speed δU.

(a) Starting from the dimensional incompressible Navier–Stokes equations, show that pressures of $O(\mu U/\delta^2 L)$ are generated on a time scale L/U as long as $\delta^2 \rho L U/\mu \ll 1$. Show further that, under these circumstances and in dimensional variables, the flow satisfies approximately the lubrication equations

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0, \quad \mu\frac{\partial^2 u}{\partial z^2} = \frac{\partial p}{\partial x}, \quad \mu\frac{\partial^2 v}{\partial z^2} = \frac{\partial p}{\partial y}, \quad \frac{\partial p}{\partial z} = 0,$$

with $u = v = w = 0$ on $z = 0$ and $u = v = 0$, $w = \mathrm{d}h/\mathrm{d}t$ on $z = h(t)$.
Hence show that

$$\frac{\mathrm{d}h}{\mathrm{d}t} + \nabla \cdot (h\bar{\mathbf{u}}) = 0,$$

where the mean in-plane velocity is

$$\bar{\mathbf{u}} = -\frac{h^2}{12\mu}\nabla p$$

and $\nabla = \left(\dfrac{\partial}{\partial x}, \dfrac{\partial}{\partial y}\right)$.

(b) The fluid is contained within a circle of radius $R(t)$, which is centred at the origin and surrounded by air of negligibly small density and viscosity. Assuming that the flow is axisymmetric, the pressure is bounded and

$$p = 0, \quad \bar{\mathbf{u}} \cdot \mathbf{e}_r = \frac{\mathrm{d}R}{\mathrm{d}t} \quad \text{on} \quad r = R(t),$$

where $r^2 = x^2 + y^2$ and $\mathbf{e}_r$ is the unit radial vector, show that

$$p = \frac{3\mu\left(r^2 - R^2\right)}{h^3}\frac{\mathrm{d}h}{\mathrm{d}t}.$$

Deduce that

$$\pi R(t)^2 h(t) = \pi R(0)^2 h(0).$$

What is the physical significance of this expression?
Do you expect this solution to be valid for both $\mathrm{d}h/\mathrm{d}t < 0$ and $\mathrm{d}h/\mathrm{d}t > 0$?

B6 a 2008

4. (i) A viscous fluid fills a two-dimensional slider bearing of length L between the surfaces $y = 0$ and $y = h_0 h(\frac{x}{L})$, where $h(0) = 1$. The wall $y = 0$ is moving with constant velocity U in the x direction and the pressure in the fluid at $x = 0$ and $x = L$ is a constant p_0. If $\frac{h_0}{L}$ and $\frac{Uh_0^2}{\nu L}$ are small quantities, show that the pressure p satisfies the equation

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[h^3 \frac{\mathrm{d}p}{\mathrm{d}x} - 6h\right] = 0,$$

where both p and x are suitable nondimensional variables.

(ii) If $h(x) = \mathrm{e}^{-\alpha x}$, show that if $\alpha > 0$, the pressure within the bearing first increases with x and then decreases, and that the maximum pressure occurs at the point

$$x = x_m = -\frac{1}{\alpha} \log\left(\frac{3\left(\mathrm{e}^{2\alpha} - 1\right)}{2\left(\mathrm{e}^{3\alpha} - 1\right)}\right).$$

Show that if $\alpha \ll 1$, then $x_m \simeq \frac{1}{2}$.

B6 a 2007

3. In a Hele-Shaw cell, a viscous fluid is injected with velocity U between two rigid parallel plates which are of lateral extent L and a fixed distance h apart. Starting from the Navier–Stokes equations with the z-axis normal to the plates, show that, provided $\dfrac{Uh^2}{\nu L} \lll 1$ and $\dfrac{h}{L} \lll 1$, the flow satisfies the approximate equations

$$\begin{aligned} 0 &= -\frac{\partial p}{\partial x} + \mu \frac{\partial^2 u}{\partial z^2}, \\ 0 &= -\frac{\partial p}{\partial y} + \mu \frac{\partial^2 v}{\partial z^2}, \\ 0 &= -\frac{\partial p}{\partial z}, \end{aligned}$$

and

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0,$$

with $u = v = w = 0$ on $z = 0$ and $z = h$.

Hence show that, if $\bar{u}$, $\bar{v}$ are the *mean* velocities in the x and y directions respectively, then

$$(\bar{u}, \bar{v}) = -\frac{h^2}{12\mu} \nabla p \quad \text{and} \quad \nabla^2 p = 0,$$

where $\nabla = \left(\dfrac{\partial}{\partial x}, \dfrac{\partial}{\partial y} \right)$.

A circular blob of fluid of radius R_0 centred at the origin is at rest within the cell. At $t = 0$, a source of constant strength Q is introduced at the origin so that in the subsequent flow the fluid is contained within a circle of radius $R(t)$. What are the boundary conditions on p and $\bar{\mathbf{u}} = (\bar{u}, \bar{v})$ on $r = R(t)$?

Show that a possible solution gives

$$R(t) = \sqrt{R_0^2 + \frac{Qt}{\pi h}},$$

and determine the corresponding pressure. Do you expect this solution to be valid for both $Q > 0$ and $Q < 0$?

B6 a 2006

3. A thin two-dimensional sheet of viscous fluid lies on a horizontal table. Initially the sheet is of width L and maximum height δL where $\delta \ll 1$, and the fluid flows over the table under gravity. By non-dimensionalising the Navier-Stokes equations appropriately, show that the horizontal velocity is of $O\left(\frac{\delta^3 L^2 g}{\nu}\right)$, and that, provided $\frac{\delta^5 L^3 g}{\nu^2} \ll 1$, the equations can be approximated to

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0,$$

$$\frac{\partial p}{\partial y} + 1 = 0,$$

$$\frac{\partial p}{\partial x} - \frac{\partial^2 u}{\partial y^2} = 0,$$

where the variables have all been scaled appropriately, and x, y are the horizontal and vertical axes, u, v are the horizontal and vertical velocity components, and p is the pressure.

Given that the stress tensor $\sigma_{ij} = -p\delta_{ij} + \mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$ in dimensional coordinates, show that using the same scalings and approximations as before,

$$\sigma_{11} = \sigma_{22} = -\rho L g \delta p,$$

$$\sigma_{12} = \sigma_{21} = \rho L g \delta^2 \frac{\partial u}{\partial y}.$$

If the surface of the fluid is given by $y = h(x,t)$, show that appropriate boundary conditions are

$$u = v = 0 \quad \text{on} \quad y = 0,$$

and

$$p = \frac{\partial u}{\partial y} = 0, \quad v = \frac{\partial h}{\partial t} + u\frac{\partial h}{\partial x} \quad \text{on} \quad y = h.$$

Hence show that h satisfies the equation

$$\frac{\partial h}{\partial t} = \frac{1}{3}\frac{\partial}{\partial x}\left(h^3 \frac{\partial h}{\partial x}\right).$$

B6a 2005

3. In a slider bearing, fluid is contained between a fixed boundary at $y = h_0 h\left(\frac{x}{L}\right)$ and a boundary at $y = 0$ which moves in the x-direction with speed U. The bearing is of length L and the fluid outside the bearing is at an ambient pressure p_0. Starting from the steady Navier Stokes Equations in the usual notation:

$$(\mathbf{u}.\nabla)\,\mathbf{u} = -\frac{1}{\rho}\nabla p + \nu\nabla^2\mathbf{u}, \quad \nabla.\mathbf{u} = 0,$$

show that if $h_0 \ll L$ and $Uh_0^2 \ll \nu L$, then the problem is approximately reduced to

$$\frac{\partial^2 u}{\partial y^2} = \frac{\mathrm{d}p}{\mathrm{d}x}, \qquad \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

in terms of suitably defined nondimensional variables. What are the boundary conditions?

If $h(x) = (1+x)^{-1}$ for $0 < x < 1$, show that

$$\frac{\mathrm{d}p}{\mathrm{d}x} = 6(1+x)^2 - 12q(1+x)^3$$

where q is a constant. Show that $q = 14/45$ and sketch p as a function of x. Where does the maximum value of the pressure occur?

B6 a 2005

4. Incompressible fluid of viscosity μ flows in a saturated porous medium of permeability k and is governed by Darcy's law

$$\mathbf{u} = -\frac{k}{\mu}\nabla p$$

where p is the mean pressure and $\mathbf{u}$ is the mean velocity in the fluid. If ϕ is the void fraction of the porous medium, show that

$$\nabla.(\phi\mathbf{u}) = 0.$$

A uniform porous medium contains a region of saturated flow which is separated from a dry region by an interface $f(x, y, z, t) = 0$. Show that at the interface

$$p = p_0 \qquad \text{and} \qquad \frac{\partial f}{\partial t} + \mathbf{u}.\nabla f = 0$$

where p_0 is the constant pressure in the dry region.

The interface $x = Vt$ separates saturated flow in $x < Vt$ from a dry region in $x > Vt$. Show that the pressure in the fluid is

$$p_0 - \frac{V\mu}{k}(x - Vt).$$

By considering small disturbances to the interface of the form

$$x = Vt + \varepsilon e^{i w t + i\lambda y}$$

where λ is real, show that $w = iV|\lambda|$ and hence deduce that the interface is only stable if $V > 0$.

B6a 2004

2. A two-dimensional viscous fluid layer flows under the action of gravity on a flat plate inclined at an angle δ to the horizontal. Show that, written in terms of coordinates (x, y) tangential and normal to the plate respectively (where x is measured up the plate along the line of greatest slope), in the usual slow-flow scalings the Navier-Stokes equations are

$$\mathrm{Re}\left(\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u}\cdot\nabla)\mathbf{u}\right) = -\nabla p + \nabla^2\mathbf{u} - \left(\frac{\rho g L^2}{\mu U}\right)(\sin\delta, \cos\delta), \qquad \nabla\cdot\mathbf{u} = 0,$$

where U is the velocity scale, L is the length scale, and Re is the Reynolds number, which you should define.

Now suppose that the (dimensional) thickness of the fluid layer is $y = \delta L h(x,t)$, and that $\delta \ll 1$. Show that, in the lubrication approximation, the correct choice of the velocity scale is $U = \rho g L^2 \delta^3/\mu$, and that

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \qquad 0 = -\frac{\partial p}{\partial x} + \frac{\partial^2 u}{\partial y^2} - 1, \qquad 0 = -\frac{\partial p}{\partial y} - 1,$$

with

$$u = v = 0 \quad \text{on} \quad y = 0,$$
$$v = \frac{\partial h}{\partial t} + u\frac{\partial h}{\partial x}, \quad p = \frac{\partial u}{\partial y} = 0 \quad \text{on} \quad y = h,$$

where y, v and the pressure p have been suitably scaled. Hence show that

$$\frac{\partial h}{\partial t} = \frac{\partial}{\partial x}\left\{\frac{h^3}{3}\left(1 + \frac{\partial h}{\partial x}\right)\right\}.$$

B6a 2003

4. Under what circumstances can *lubrication theory* be used to describe the flow of a viscous fluid?

A thin two-dimensional viscous fluid layer, of thickness $h(x,t)$, flows under the action of gravity on a horizontal base $y = 0$. Show that, under the lubrication approximation, the velocity (u, v) and pressure p satisfy

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \quad 0 = -\frac{\partial p}{\partial x} + \mu\frac{\partial^2 u}{\partial y^2}, \quad 0 = -\frac{\partial p}{\partial y} - \rho g,$$

$$\text{with} \quad u = v = 0 \text{ on } y = 0, \quad v = \frac{\partial h}{\partial t} + u\frac{\partial h}{\partial x} \text{ on } y = h,$$

where μ and ρ are the constant viscosity and density of the fluid. Explain why the stress tensor is approximately

$$\begin{pmatrix} -p & \mu\dfrac{\partial u}{\partial y} \\ \mu\dfrac{\partial u}{\partial y} & -p \end{pmatrix}$$

and deduce the approximate boundary conditions $p = \partial u/\partial y = 0$ on $y = h$. Hence show that

$$\frac{\partial h}{\partial t} = \frac{\rho g}{3\mu}\frac{\partial}{\partial x}\left(h^3\frac{\partial h}{\partial x}\right).$$

4. Viscous fluid is pumped through a gap of thickness $O(h_0)$ between a stationary plate $z = 0$ and a stationary rigid surface $z = h(x)$, $0 < x < L$. Starting from the slow flow equations, and assuming the flow is two-dimensional and that $h_0 \ll L$, show that the pressure p satisfies

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(h^3\frac{\mathrm{d}p}{\mathrm{d}x}\right) = 0,$$

where, in suitable dimensionless variables, the mean velocity in the gap is

$$-\frac{h^3}{12}\frac{\mathrm{d}p}{\mathrm{d}x}.$$

Generalise this result to the case when the gap is given by $0 < z < h(x, y)$ to show that

$$\frac{\partial}{\partial x}\left(h^3\frac{\partial p}{\partial x}\right) + \frac{\partial}{\partial y}\left(h^3\frac{\partial p}{\partial y}\right) = 0$$

where the mean velocity is now

$$\frac{-h^3}{12}\nabla p.$$

Now suppose $h = 1$ and that there is a free surface separating fluid in $y < \eta(x, t)$ from a vacuum in $y > \eta(x, t)$. Show that, at $y = \eta$,

$$p = 0 \text{ and } \frac{\partial \eta}{\partial t} - \frac{1}{12}\frac{\partial p}{\partial x}\frac{\partial \eta}{\partial x} = -\frac{1}{12}\frac{\partial p}{\partial y}.$$

Hence show that it is possible for η to equal Vt if $p = -12V(y - Vt)$. How do you expect the stability of this flow to depend on the sign of V? Indicate briefly how you could verify this assertion mathematically.

3. Explain *briefly* the difference between *laminar* and *turbulent* flow.

A turbulent flow has velocity components $u_i = \bar{u}_i + u'_i$, where $\bar{u}_i$ denotes the mean component, and u'_i its fluctuation. Write down the incompressible Navier-Stokes equations, and hence derive an equation describing the evolution of $\bar{u}_i$.

Deduce that the viscous stresses are supplemented in the mean flow equation by the Reynolds stress $-\rho\overline{u'_i u'_j}$.

A constant pressure gradient $-G$ drives a turbulent flow down a pipe of radius a. Assume that the average flow is purely axial, that the dynamic viscosity is negligible, and that the Reynolds stresses can be described using an eddy viscosity ε_T. Assume also that if the time averaged axial flow is $u(r)$, then the axial component of the divergence of the Reynolds stress tensor is $\frac{1}{r}\frac{\partial}{\partial r}\left[\varepsilon_T r \frac{\partial u}{\partial r}\right]$.

If the maximum averaged velocity is u_m at $r = 0$, and the eddy viscosity is taken to be of the form

$$\varepsilon_T = \rho u_m a \phi(r/a),$$

show that

$$u_m = \left[\frac{Gac}{\rho}\right]^{1/2},$$

and give an expression for the constant c.

4. Viscous fluid with a constant kinematic viscosity ν occupies the gap, of thickness $O(h_0)$, between a flat plate $\hat{z} = 0$ moving with constant velocity U in the $\hat{x}$-direction and a stationary rigid surface described by $\hat{z} = h_0 h(\hat{x}/L)$, $0 < \hat{x} < L$. The pressure in the surrounding fluid is p_0. From the Navier-Stokes equations, taking the fluid flow to be two-dimensional, assuming that both h_0/L and the reduced Reynolds number $Uh_0^2/\nu L$ are small and that non-dimensional coordinates are $x = \hat{x}/L, z = \hat{z}/h_0$, show that a suitable scale for the pressure, $\hat{p}$, is

$$\hat{p} - p_0 = \frac{\mu U L}{h_0^2} p.$$

Hence deduce that the pressure, p, satisfies the equation

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(h^3 \frac{\mathrm{d}p}{\mathrm{d}x}\right) = 6\frac{\mathrm{d}h}{\mathrm{d}x},$$

with $p = 0$ at $x = 0$ and $x = 1$.

A slider bearing consists of a flat plate $z = 0$ moving with constant velocity U in the x-direction and a fixed slider at a dimensionless height $h(x) = (1 - \lambda x)$ for $0 < x < 1$ and where $0 < \lambda < 1$. Show that the dimensionless fluid flux through the gap is $(1-\lambda)/(2-\lambda)$, and hence show that the scaled pressure within the bearing is

$$\frac{6\lambda x(1-x)}{(2-\lambda)(1-\lambda x)^2}.$$

B6a 2000

3. Two parallel discs of radius a are separated by a thin layer of viscous fluid. If the thickness of the layer is $h(t)$, show that under certain conditions which you should state clearly, the pressure p in the fluid satisfies the equation

$$\frac{h^3}{12\mu}\left(\frac{\partial^2 p}{\partial x^2}+\frac{\partial^2 p}{\partial y^2}\right)=\frac{dh}{dt}$$

where μ is the coefficient of viscosity and x, y are coordinates in the plane of the discs.

If the pressure outside the layer is a constant p_0, show that

$$p=p_0-\frac{3\mu}{h^3}(a^2-r^2)\frac{dh}{dt}$$

where $r^2=x^2+y^2$. Hence show that the outwards force that must be exerted on each disc to separate them is

$$\frac{3\pi\mu a^4}{2h^3}\frac{dh}{dt}.$$

The discs are pulled apart at a constant rate, so that $h(t)=h_0(1+\lambda t)$ where $h_0 \ll a$ and $\lambda > 0$. Show that the reduced Reynolds number is $\frac{h\dot{h}}{\nu} \sim \frac{h_0^2\lambda^2 t}{\nu}$ as $t \to \infty$. Hence show that the above theory is valid for

$$t<\min\left(\frac{a}{h_0\lambda},\frac{\nu}{h_0^2\lambda^2}\right).$$

B6a 1999

3. (i) A thin layer of viscous fluid flows down a vertical wall $y = 0$, $x \geqslant 0$ under the action of gravity. Starting from the slow flow equations, state the assumption made in deriving, in the usual notation, the following lubrication model for the film velocity (u, v) and thickness h:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \quad 0 = -\frac{\partial p}{\partial x} + \rho g + \mu\frac{\partial^2 u}{\partial y^2}, \quad 0 = -\frac{\partial p}{\partial y}$$

with

$$u = v = 0 \text{ on } y = 0; \quad p = 0, \ \frac{\partial u}{\partial y} = 0, \ v = \frac{\partial h}{\partial t} + u\frac{\partial h}{\partial x} \quad \text{on } y = h.$$

Deduce that

$$\frac{\partial h}{\partial t} + \rho\frac{gh^2}{\mu}\frac{\partial h}{\partial x} = 0.$$

(ii) A thin layer of viscous fluid flows along a horizontal plane $y = 0$ under gravity. Show that similar assumptions lead to the model

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \quad 0 = -\frac{\partial p}{\partial x} + \mu\frac{\partial^2 u}{\partial y^2}, \quad 0 = -\frac{\partial p}{\partial y} - \rho g$$

with the same boundary conditions.

Deduce that

$$\frac{\partial h}{\partial t} = \frac{\rho g}{3\mu}\frac{\partial}{\partial x}\left(h^3\frac{\partial h}{\partial x}\right).$$

(iii) Show that if $h = h_0(x)$ at $t = 0$, the solution of the model in (i) is $h = h_0(x - \frac{gh^2 t}{\nu})$ and sketch the solution when $h_0 = e^{-x^2}$. What would happen to this initial profile in (ii)?

Turn Over

4. A viscous fluid is forced to flow with a velocity of order U through a gap whose size is of order H between two plates whose length and breadth are of order L. Let x, y, z be non-dimensional coordinates scaled by L, L, H respectively, with x, y in the plane of one plate at $z = 0$; the other plate is given by

$$z = h(x, y).$$

Assuming the slow flow equations, show that if $H/L \ll 1$, then to leading order, the pressure p satisfies

$$p \sim p_0(x, y),$$

and that the average velocity in the (x, y) plane is proportional to $-\nabla p_0$. Show further that p_0 satisfies

$$\frac{\partial}{\partial x}(h^3 \frac{\partial p_0}{\partial x}) + \frac{\partial}{\partial y}(h^3 \frac{\partial p_0}{\partial y}) = 0.$$

Deduce that if h is constant, then $\nabla^2 p_0 = 0$, and show that if, in this case, the flow is produced by a source placed between the plates, then

$$p_0 \sim -\frac{Q}{2\pi} \log r,$$

near the source, Q being a measure of its strength and $r = \sqrt{x^2 + y^2}$. Show further that the velocity field is the same as that produced by a line source in an irrotational inviscid flow. What is the pressure in the latter case?