#3

B6a 2008

3. You are given that the steady Navier-Stokes equations in plane polar co-ordinates are

$$\begin{aligned}
\frac{1}{r}\frac{\partial}{\partial r}(ru_r) + \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} &= 0, \\
u_r\frac{\partial u_r}{\partial r} + \frac{u_\theta}{r}\frac{\partial u_r}{\partial \theta} - \frac{u_\theta^2}{r} &= -\frac{1}{\rho}\frac{\partial p}{\partial r} + \nu\left(\nabla^2 u_r - \frac{u_r}{r^2} - \frac{2}{r^2}\frac{\partial u_\theta}{\partial \theta}\right), \\
u_r\frac{\partial u_\theta}{\partial r} + \frac{u_\theta}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{u_r u_\theta}{r} &= -\frac{1}{\rho r}\frac{\partial p}{\partial \theta} + \nu\left(\nabla^2 u_\theta + \frac{2}{r^2}\frac{\partial u_r}{\partial \theta} - \frac{u_\theta}{r^2}\right),
\end{aligned}$$

where $\nabla^2 = \dfrac{\partial^2}{\partial r^2} + \dfrac{1}{r}\dfrac{\partial}{\partial r} + \dfrac{1}{r^2}\dfrac{\partial^2}{\partial \theta^2}$.

(i) Show that if the flow is purely radial then

$$u_r = \frac{g(\theta)}{r},$$

where $g(\theta)$ satisfies the equation

$$\nu(g''' + 4g') + 2gg' = 0.$$

(ii) Flow is generated in a wedge $-\alpha < \theta < \alpha$ by a source of strength Q at the origin. Assuming that the flow in the wedge is radial, write down the boundary conditions for g and hence show that an appropriate nondimensionalization is $g = QG(\theta)$. If $\frac{Q}{\nu} = \epsilon$ is small, write $G = G_0 + \epsilon G_1 + \cdots$ and show that G_0 satisfies

$$G_0''' + 4G_0' = 0,$$

with $G_0(\pm\alpha) = 0$ and $\int_{-\alpha}^{\alpha} G_0(\theta)\mathrm{d}\theta = 1$. Hence show that $G_0 = C(\cos 2\theta - \cos 2\alpha)$ and determine the constant C.

(iii) Show that the expansion for G will break down as α approaches a critical value α_c defined by

$$2\alpha_c = \tan 2\alpha_c.$$

Sketch the function $G(\theta)$ for α in the interval $(\frac{\pi}{2}, \alpha_c)$ and explain (without calculations) what happens to the solution as $\alpha \to \alpha_c$.

B6a 2007

4. The Stokes equations for slow flow are, in dimensionless form,

$$\nabla.\mathbf{u} = 0, \qquad \nabla p = \nabla^2 \mathbf{u}.$$

Show that for two-dimensional flow there exists a stream function ψ which satisfies the biharmonic equation

$$\nabla^4 \psi = 0.$$

A uniform stream, velocity $U\mathbf{i}$, flows past a circular cylinder of radius a. Show that the boundary conditions for the stream function above are

$$\psi = 0, \qquad \frac{\partial \psi}{\partial r} = 0 \text{ on } r = 1,$$

and $\psi \to r \sin\theta$ as $r \to \infty$.

Given that $\nabla^2 = \dfrac{\partial^2}{\partial r^2} + \dfrac{1}{r}\dfrac{\partial}{\partial r} + \dfrac{1}{r^2}\dfrac{\partial^2}{\partial \theta^2}$ in plane polar co-ordinates, show that $\psi = f(r)\sin\theta$, where

$$\left(\frac{\mathrm{d}^2}{\mathrm{d}r^2} + \frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r} - \frac{1}{r^2}\right)^2 f = 0 \qquad (1)$$

with

$$f(1) = f'(1) = 0 \text{ and } f' \to 1 \text{ as } r \to \infty. \qquad (2)$$

Show that the general solution of (1) is

$$f = Ar^3 + Br\log r + Cr + Dr^{-1},$$

and that it is impossible to satisfy conditions (2) with this solution.

Explain why there is no solution to this problem, and indicate what you would have to do in order to solve the problem of flow past a circular cylinder.

B6 a 2006

2. Starting from the non-dimensional Navier-Stokes equations given in question 1, derive the slow flow approximation

$$(\text{curl})^3\mathbf{u} = 0, \qquad \nabla.\mathbf{u} = 0,$$

when $Re \ll 1$.

You are given that in axisymmetric flow in spherical polar coordinates, these equations are equivalent to

$$\left(\frac{\partial^2}{\partial r^2} - \frac{\cot\theta}{r^2}\frac{\partial}{\partial\theta} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right)^2 \psi = 0,$$

where $\mathbf{u} = \dfrac{1}{r^2\sin\theta}\dfrac{\partial\psi}{\partial\theta}\mathbf{e}_r - \dfrac{1}{r\sin\theta}\dfrac{\partial\psi}{\partial r}\mathbf{e}_\theta$. Show that there is a solution of these equations of the form $\psi = f(r)\sin^2\theta$ where $f(r) = Ar^4 + Br^2 + Cr + Dr^{-1}$. Hence find the velocity field for the flow of a uniform stream $\mathbf{i}$ past a sphere of radius unity centred at the origin, and show that $\mathbf{u} = \mathbf{i} + O\left(\dfrac{1}{r}\right)$ as $r \to \infty$.

Explain the difficulty that arises when one tries to solve the slow flow equations for a uniform flow past a *circular cylinder* by a similar method to that used above, and describe, in general terms, the resolution of this difficulty.

06

B6a 2004

4. A steady uniform stream $U\mathbf{i}$ of an incompressible fluid of dynamic viscosity μ flows past a circular cylinder of radius a whose axis is parallel to the unit vector $\mathbf{k}$. Show that for slow flow an appropriate nondimensionalisation of the Navier-Stokes equations leads to

$$\mathrm{Re}(\mathbf{u}\cdot\nabla)\mathbf{u} = -\nabla p + \nabla^2\mathbf{u}, \qquad \nabla\cdot\mathbf{u} = 0,$$

where $\mathbf{u}$ and p are the nondimensional fluid velocity and pressure respectively, and Re is the Reynolds number, which you should define. What are the boundary conditions on $\mathbf{u}$? If the flow is two-dimensional in a plane perpendicular to $\mathbf{k}$, show that there is a streamfunction ψ such that $\mathbf{u} = \mathrm{curl}(\psi\mathbf{k})$. What are the boundary conditions which must be imposed on ψ?

If $\mathrm{Re} \ll 1$ show that the slow flow approximation to the Navier-Stokes equations leads to

$$\nabla^4\psi = 0.$$

By separating the variables as $\psi = f(r)\sin\theta$ show that

$$f = \frac{A}{r} + Br + Cr\log r + Dr^3.$$

Show that it is not possible for this solution to satisfy all the boundary conditions on ψ. Explain, without detailed calculations, how this paradox is resolved. Given that the resolution of the paradox leads to $C = -1/\log\mathrm{Re}$, $D = 0$, use the remaining boundary conditions to determine A and B.

B6·a 2003

3. The steady dimensionless Navier–Stokes equations in plane polar coordinates (r,θ) are

$$\begin{aligned}
\frac{1}{r}\frac{\partial}{\partial r}(ru_r)+\frac{1}{r}\frac{\partial u_\theta}{\partial \theta} &= 0,\\
u_r\frac{\partial u_r}{\partial r}+\frac{u_\theta}{r}\frac{\partial u_r}{\partial \theta}-\frac{u_\theta^2}{r} &= -\frac{\partial p}{\partial r}+\frac{1}{\mathrm{Re}}\left(\nabla^2 u_r-\frac{u_r}{r^2}-\frac{2}{r^2}\frac{\partial u_\theta}{\partial \theta}\right),\\
u_r\frac{\partial u_\theta}{\partial r}+\frac{u_\theta}{r}\frac{\partial u_\theta}{\partial \theta}+\frac{u_r u_\theta}{r} &= -\frac{1}{r}\frac{\partial p}{\partial \theta}+\frac{1}{\mathrm{Re}}\left(\nabla^2 u_\theta+\frac{2}{r^2}\frac{\partial u_r}{\partial \theta}-\frac{u_\theta}{r^2}\right),
\end{aligned}$$

where

$$\nabla^2\phi=\frac{\partial^2\phi}{\partial r^2}+\frac{1}{r}\frac{\partial \phi}{\partial r}+\frac{1}{r^2}\frac{\partial^2\phi}{\partial \theta^2}$$

and Re is the Reynolds number. Show that a solution exists in which $u_\theta = 0$ and $u_r = g(\theta)/r$, provided

$$\frac{1}{\mathrm{Re}}\left(\frac{\mathrm{d}^3 g}{\mathrm{d}\theta^3}+4\frac{\mathrm{d}g}{\mathrm{d}\theta}\right)+2g\frac{\mathrm{d}g}{\mathrm{d}\theta}=0.$$

Such a flow is driven by a source at the origin in the wedge $-\alpha < \theta < \alpha$, and the nondimensional fluid flux is Q. Show that, as $\mathrm{Re}\to\infty$, $g\to Q/2\alpha$ except near $\theta=\pm\alpha$. Assume that, in this limit, there is a boundary layer near $\theta=-\alpha$. Hence show, to lowest order in the boundary layer, that

$$\frac{\mathrm{d}^2 g}{\mathrm{d}\phi^2}+g^2=\frac{Q^2}{4\alpha^2},$$

where $\phi=\sqrt{\mathrm{Re}}(\theta+\alpha)$, that $g=0$ on $\phi=0$ and that $g\to Q/2\alpha$ as $\phi\to\infty$. Deduce that this boundary layer can only exist if $Q<0$ and comment on the implications for "sucking out" a candle flame.

B6a 2002

3. (i) Write down the solution of the problem

$$\frac{d^2y}{dx^2} + \epsilon\frac{dy}{dx} = 0 \quad \text{for } 0 < x < L$$

with boundary conditions $y(0) = 0$ and $y(L) = 1$. Expand this solution asymptotically as $\epsilon \downarrow 0$ to show that

$$y = \frac{x}{L} + \frac{\epsilon}{2L}x(L-x) + O(\epsilon^2).$$

This formula gives $y'(0) \sim \frac{\epsilon}{2}$ as $L \to \infty$; explain why this result is a bad approximation for sufficiently large L.

(ii) Under what conditions are the dimensionless *slow flow* equations

$$0 = -\nabla p + \nabla^2\mathbf{u}, \quad \nabla\cdot\mathbf{u} = 0$$

expected to be a valid model for viscous incompressible flow for a stream of velocity U_0 past an obstacle of size a? Show that in two-dimensional steady flow with $\mathbf{u} = (u(x,y), v(x,y), 0)$, the slow flow equations imply that the stream function ψ satisfies $\nabla^4\psi = 0$.

Assume that for two-dimensional flow in polar coordinates (r,θ) these equations allow the stream function to be written as $\psi = f(r)\sin\theta$ where

$$\left(\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} - \frac{1}{r^2}\right)^2 f = 0.$$

For flow past a circular cylinder $r = a$, write down the boundary conditions that f must satisfy at $r = a$ and show that, if these conditions are satisfied, then ψ cannot represent a uniform free stream as $r \to \infty$. Suggest how this difficulty might be overcome.

B6a 2000

2. Show that an appropriate nondimensionalisation of the Navier-Stokes equations for the steady flow of a uniform stream $U\mathbf{i}$ past a circular cylinder of radius a leads to equations for the velocity $\mathbf{u}$ and pressure p in the form

$$\begin{aligned} \mathrm{Re}(\mathbf{u}.\nabla)\mathbf{u} &= -\nabla p + \nabla^2 \mathbf{u} \\ \nabla.\mathbf{u} &= 0. \end{aligned}$$

where $\mathrm{Re} = \frac{Ua}{\nu}$ and ν is the viscosity of the fluid.

If the flow is two-dimensional in a plane perpendicular to the unit vector $\mathbf{k}$ along the axis of the cylinder, show that there is a stream function ψ given by $\mathbf{u} = \mathrm{curl}(\psi\mathbf{k})$ and that ψ satisfies the boundary conditions

$$\psi = \frac{\partial\psi}{\partial r} = 0 \quad \text{on} \quad r = 1 \quad \text{and} \quad \psi \sim r\sin\theta \quad \text{as} \quad r \to \infty.$$

If $\mathrm{Re} \ll 1$, show that the slow flow approximation to the Navier–Stokes equations can be written in the form

$$\mathrm{curl}^4(\psi\mathbf{k}) = \mathbf{0}.$$

Given that $\psi = f(r)\sin\theta$ satisfies this equation if $f = Ar^3 + Br + Cr\log r + Dr^{-1}$ with A, B, C, D constant, show that it is not possible for this solution to satisfy all the required boundary conditions on ψ.

Explain, without detailed calculations, what has to be done to find a uniformly valid approximation for ψ.

B6a 1999

4. Starting from the two-dimensional Navier-Stokes equation in the form

$$\frac{\partial \omega}{\partial t} + \frac{\partial(\psi,\omega)}{\partial(y,x)} = \frac{1}{Re}\nabla^2\omega, \quad \omega = -\nabla^2\psi$$

where ω is the vorticity, ψ is the stream function and Re is the Reynolds Number, show that a nearly unidirectional flow is possible in which

$$\psi \sim \Re\Big(\psi_0(y) + \varepsilon g(y) e^{ik(x-ct)}\Big),$$

where $\Re$ means 'real part', as long as

$$\frac{1}{Re}(g'''' - 2k^2 g'' + k^4 g) = -ik\psi_0''' g + ik(\psi_0' - c)(g'' - k^2 g).$$

Show that if $Re = \infty$ and $\psi =$ const. on $y = 0$ and on $y = 1$, then

$$\int_0^1 \frac{\psi_0'''(y)|g|^2\,dy}{\psi_0' - c} = -\int_0^1 (|g'|^2 + k^2|g|^2)dy$$

as long as $\psi_0' \neq c$ in $0 < y < 1$. Taking k to be real, equate imaginary parts to show that instability is only possible if the velocity profile corresponding to ψ_0 has a point of inflection.

B6a 1998

Fluid Dynamics

1. The Navier-Stokes equations describing two-dimensional incompressible flow can be written in the form

$$\rho\left[\frac{\partial(\nabla^2\psi)}{\partial t}+\frac{\partial(\psi,\nabla^2\psi)}{\partial(y,x)}\right]=\mu\nabla^4\psi,$$

where ψ is the stream function. When the characteristic velocity and length are U and L respectively, show that the dimensionless stream function satisfies

$$\frac{\partial(\nabla^2\psi)}{\partial t}+\frac{\partial(\psi,\nabla^2\psi)}{\partial(y,x)}=\frac{1}{Re}\nabla^4\psi, \qquad (*)$$

where $Re=\dfrac{\rho UL}{\mu}$ and ψ, x, y and t are now dimensionless.

Fluid flows steadily in a two-dimensional wedge defined by $-\alpha<\theta<\alpha$ in polar coordinates under the action of a source at the origin whose volume flow rate (per unit width transverse to the flow) is M. Show that a suitable scaling for the velocity at distance L from the source is $M/2\alpha L$ and that $Re = M\rho/2\alpha\mu$. Show that a possible flow is $\psi=f(\theta)$ where

$$f''''+4f''+2Ref'f''=0$$

with $f'(\pm\alpha)=0$, $f(\alpha)-f(-\alpha)=2\alpha$.

Find an approximate solution for $v=f'(\theta)$ when $Re\ll 1$, and sketch the graph of $v(\theta)$. How do you expect the profile for v to be modified if $Re\gg 1$?

[*In plane polar coordinates* $\nabla^2=\dfrac{\partial^2}{\partial r^2}+\dfrac{1}{r}\dfrac{\partial}{\partial r}+\dfrac{1}{r^2}\dfrac{\partial^2}{\partial\theta^2}$.]

B6a 1998

3. Assuming that slow viscous flow can be modelled by

$$\mathbf{0} = -\nabla p + \mu \nabla^2 \mathbf{u}, \ \ \nabla . \mathbf{u} = 0,$$

where $\mathbf{u}$ is the velocity, p is the pressure, and μ is the viscosity, show that

$$\text{curl}^3 \mathbf{u} = \mathbf{0}.$$

Assuming further that the continuity equation is identically satisfied if

$$\mathbf{u} = \text{curl}\left(0,\ 0,\ \frac{\psi}{r\sin\theta}\right) = \left(\frac{1}{r^2\sin\theta}\frac{\partial\psi}{\partial\theta},\ -\frac{1}{r\sin\theta}\frac{\partial\psi}{\partial r},\ 0\right)$$

in spherical polar coordinates $(r,\ \theta,\ \phi)$, and given that

$$\text{curl}^2\left(0,\ 0,\ \frac{\psi}{r\sin\theta}\right) = \left(0,\ 0,\ -\frac{D^2\psi}{r\sin\theta}\right),$$

where

$$D^2 = \frac{\partial^2}{\partial r^2} - \frac{\cot\theta}{r^2}\frac{\partial}{\partial\theta} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2},$$

show that the stream function ψ satisfies $D^4\psi = 0$.

Write down the boundary conditions when a fluid with undisturbed velocity U streams past

(a) a rigid impermeable sphere $r = a$;

(b) a bubble containing air at pressure p_0 whose surface tension is so large that the bubble surface is also $r = a$.

Find the solution in the form $\psi = f(r)\sin^2\theta$ in each case.

[*You may assume the stress tensor is such that* $\sigma_{r\theta} = \mu\left[r\frac{\partial}{\partial r}\left(\frac{1}{r}u_\theta\right) + \frac{1}{r}\frac{\partial u_r}{\partial\theta}\right]$.]

Bb a 1997

4. A viscous fluid is forced to flow with a velocity of order U through a gap whose size is of order H between two plates whose length and breadth are of order L. Let x, y, z be non-dimensional coordinates scaled by L, L, H respectively, with x, y in the plane of one plate at $z = 0$; the other plate is given by

$$z = h(x, y).$$

Assuming the slow flow equations, show that if $H/L \ll 1$, then to leading order, the pressure p satisfies

$$p \sim p_0(x, y),$$

and that the average velocity in the (x, y) plane is proportional to $-\nabla p_0$. Show further that p_0 satisfies

$$\frac{\partial}{\partial x}(h^3 \frac{\partial p_0}{\partial x}) + \frac{\partial}{\partial y}(h^3 \frac{\partial p_0}{\partial y}) = 0.$$

Deduce that if h is constant, then $\nabla^2 p_0 = 0$, and show that if, in this case, the flow is produced by a source placed between the plates, then

$$p_0 \sim -\frac{Q}{2\pi} \log r,$$

near the source, Q being a measure of its strength and $r = \sqrt{x^2 + y^2}$. Show further that the velocity field is the same as that produced by a line source in an irrotational inviscid flow. What is the pressure in the latter case?