B6.a. 1997

2. A flat plate of length L is aligned with a free stream flow $U\mathbf{i}$. Starting from the Navier-Stokes equations in the form

$$\nabla.\mathbf{u} = 0, \quad (\mathbf{u}.\nabla)\mathbf{u} = -\frac{1}{\rho}\nabla p + \nu\nabla^2\mathbf{u},$$

show that if $UL/\nu >> 1$, the flow near the plate can be described by the boundary layer equations

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \frac{\partial^2 u}{\partial y^2}, \quad \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \tag{2.1}$$

when written in suitable non-dimensional variables. What boundary conditions have to be imposed? *Verify* that if the plate is semi-infinite, with the leading edge at the origin, then $u = f'(\eta)$, $v = -x^{-1/2}(f(\eta) - \eta f'(\eta))/2$, with $\eta = yx^{-1/2}$ satisfy (2.1) as long as

$$f''' + \frac{1}{2}ff'' = 0,$$

with $f(0) = f'(0) = 0$ and $f' \to 1$ as $\eta \to \infty$. Explain briefly why (2.1) should reduce to an ordinary differential equation in this case.

What extra term would appear in (2.1) if the plate is replaced by a finite blunt body? Give a rough sketch of the streamlines and compare them with those predicted by an inviscid irrotational model.

B6·a. 1998

Fluid Dynamics

1. The Navier-Stokes equations describing two-dimensional incompressible flow can be written in the form

$$\rho\left[\frac{\partial(\nabla^2\psi)}{\partial t}+\frac{\partial(\psi,\nabla^2\psi)}{\partial(y,x)}\right]=\mu\nabla^4\psi,$$

where ψ is the stream function. When the characteristic velocity and length are U and L respectively, show that the dimensionless stream function satisfies

$$\frac{\partial(\nabla^2\psi)}{\partial t}+\frac{\partial(\psi,\nabla^2\psi)}{\partial(y,x)}=\frac{1}{Re}\nabla^4\psi, \qquad (*)$$

where $Re=\dfrac{\rho UL}{\mu}$ and ψ, x, y and t are now dimensionless.

Fluid flows steadily in a two-dimensional wedge defined by $-\alpha<\theta<\alpha$ in polar coordinates under the action of a source at the origin whose volume flow rate (per unit width transverse to the flow) is M. Show that a suitable scaling for the velocity at distance L from the source is $M/2\alpha L$ and that $Re=M\rho/2\alpha\mu$. Show that a possible flow is $\psi=f(\theta)$ where

$$f''''+4f''+2Ref'f''=0$$

with $f'(\pm\alpha)=0$, $f(\alpha)-f(-\alpha)=2\alpha$.

Find an approximate solution for $v=f'(\theta)$ when $Re\ll 1$, and sketch the graph of $v(\theta)$. How do you expect the profile for v to be modified if $Re\gg 1$?

[*In plane polar coordinates* $\nabla^2=\dfrac{\partial^2}{\partial r^2}+\dfrac{1}{r}\dfrac{\partial}{\partial r}+\dfrac{1}{r^2}\dfrac{\partial^2}{\partial\theta^2}$.]

B.6.a 1999

2. Incompressible fluid of kinematic viscosity ν flows past a plane wall $y = 0$ with velocity $U(x) = U_0(x/L)^m$ where U_0, L and m are constants. The stream function is ψ, which vanishes on the wall. State the conditions under which there is a boundary layer in which the dimensionless stream function Ψ satisfies

$$\frac{\partial \Psi}{\partial Y}\frac{\partial^2 \Psi}{\partial X \partial Y} - \frac{\partial \Psi}{\partial X}\frac{\partial^2 \Psi}{\partial Y^2} = \frac{\partial^3 \Psi}{\partial Y^3} + mX^{2m-1},$$

with

$$\frac{\partial \Psi}{\partial Y} \to X^m \quad \text{as } Y \to \infty, \qquad \Psi = \frac{\partial \Psi}{\partial Y} = 0 \quad \text{on } Y = 0.$$

Also, define X, Y and Ψ in terms of x, y and ψ. Show that there is a similarity solution in which

$$\Psi = X^{(m+1)/2} f(Y/X^{(1-m)/2})$$

where

$$f''' + \frac{m+1}{2} f f'' = m(f'^2 - 1)$$

with $f(0) = f'(0) = 0$, $f'(\infty) = 1$. How does the fact that this problem only has stable solutions for $m > -0.09\ldots$ relate to the theory of flight?

B6.a 2000

Viscous Flow

1. A two-dimensional flow impinges on a fixed wall, $y = 0$, so that the inviscid approximation for the velocity near the stagnation point is $(\alpha x, -\alpha y)$. Show that if the length scale L in the x-direction is chosen such that $L \gg \sqrt{\frac{\nu}{\alpha}}$ where ν is the kinematic viscosity, there will be a boundary layer on the wall of width $O\left(\sqrt{\frac{\nu}{\alpha}}\right)$ where the nondimensional equations for the stream function may be written as

$$\frac{\partial \psi}{\partial Y}\frac{\partial^2 \psi}{\partial X \partial Y} - \frac{\partial \psi}{\partial X}\frac{\partial^2 \psi}{\partial Y^2} = X + \frac{\partial^3 \psi}{\partial Y^3}$$

where X, Y are suitably scaled coordinates.

Write down the appropriate boundary conditions for ψ.

By substituting $\psi = X^\gamma f(Y/X^\beta)$ into this equation, show that such a similarity solution only exists if $\gamma = 1$, $\beta = 0$ and f satisfies

$$f''' + ff'' = f'^2 - 1,$$

with $f(0) = f'(0) = 0$ and $f'(\infty) = 1.$

B.6a 2001

2. A uniform, incompressible stream of fluid of kinematic viscosity ν and speed U flows past a flat plate of length L. Define a Reynolds number and write down a suitable dimensionless form of the Navier–Stokes equations. When the Reynolds number is large, derive the two-dimensional boundary layer equations

$$\begin{aligned} u\frac{\partial u}{\partial x} + V\frac{\partial u}{\partial Y} &= \frac{\partial^2 u}{\partial Y^2}, \\ \frac{\partial u}{\partial x} + \frac{\partial V}{\partial Y} &= 0, \end{aligned} \tag{1}$$

where the non-dimensional velocities (u, V) and coordinates (x, Y) should be defined.

What boundary conditions have to be imposed?

If ψ denotes the stream function and the leading edge of the plate is at $x = 0$, show that a solution exists of the form $\psi = \sqrt{x} f(\eta)$, where $\eta = Y/\sqrt{x}$, as long as

$$f''' + \tfrac{1}{2} f f'' = 0, \tag{2}$$

with $f(0) = f'(0) = 0$ and $f' \to 1$ as $\eta \to \infty$.

Describe how the invariance of (2) under the transformation $f(\eta) \to a f(a\eta)$ enables the boundary value problem for f to be recast as an initial value problem.

B6a 2002

2. Explain the physical significance of the terms in the energy equation

$$\rho c \frac{\mathrm{d}T}{\mathrm{d}t} = k\nabla^2 T + \Phi,$$

in the usual notation, where T is the fluid temperature. Does Φ increase or decrease with the viscosity and the strain-rate respectively? Show that in an inviscid uniform flow in the x-direction, the non-dimensional version of this equation is

$$\frac{\partial T}{\partial x} = \epsilon \nabla^2 T,$$

where ϵ is a constant.

Suppose that the ambient temperature is zero and that a plate at temperature $T = 1$ lies along $y = 0$, $x > 0$. Show that as $\epsilon \to 0$, there is a thermal boundary layer on the plate in which $Y = y/\sqrt{\epsilon} = O(1)$ and, to lowest order, T satisfies

$$\frac{\partial T}{\partial x} = \frac{\partial^2 T}{\partial Y^2}.$$

What boundary conditions must be satisfied by the solution to this equation? Show that

$$T = \frac{2}{\sqrt{\pi}} \int_{Y/2\sqrt{x}}^{\infty} \mathrm{e}^{-s^2}\,\mathrm{d}s$$

satisfies the equation and boundary conditions, and indicate on a sketch where you expect the solution to be valid.

Apply the same argument to show that if the dimensionless stream function ψ in a high Reynolds number flow past a plate lying along $y = 0$, $x > 0$ satisfies

$$\frac{\partial(\psi, \nabla^2\psi)}{\partial(y, x)} = \epsilon \nabla^4 \psi$$

with $\frac{\partial \psi}{\partial y} \to 1$ as $x^2 + y^2 \to \infty$, then, in the boundary layer,

$$\left(\frac{\partial \Psi}{\partial Y}\frac{\partial}{\partial x} - \frac{\partial \Psi}{\partial x}\frac{\partial}{\partial Y}\right)\frac{\partial^2 \Psi}{\partial Y^2} = \frac{\partial^4 \Psi}{\partial Y^4}.$$

What is the relation between ψ and Ψ? What are the boundary conditions on Ψ? Where do you expect the boundary layer approximation to be valid?

B6.a 2003 ·

2. The streamfunction in a two-dimensional flow past a flat plate $y = 0$ satisfies the dimensionless equation

$$\frac{\partial(\psi, \nabla^2\psi)}{\partial(y, x)} = \frac{1}{\mathrm{Re}} \nabla^4 \psi,$$

where Re is the Reynolds number. When $\mathrm{Re} = \infty$, there is a slip velocity $U(x)$ on the plate. Show that, when Re is large but finite, the flow near the plate only differs appreciably from $U(x)$ in a boundary layer in which $Y = y\sqrt{\mathrm{Re}} = O(1)$ and that, in this boundary layer,

$$\psi(x, y) = \frac{1}{\sqrt{\mathrm{Re}}} \Psi(x, Y), \quad \text{where} \quad \frac{\partial(\Psi, \partial^2\Psi/\partial Y^2)}{\partial(Y, x)} = \frac{\partial^4 \Psi}{\partial Y^4}.$$

As well as $\Psi(x, 0) = \partial\Psi/\partial Y(x, 0) = 0$, what other condition is needed to determine Ψ?

Suppose the solution is a similarity solution in which $\Psi(x, Y) = f(x) g\left(Y/h(x)\right)$. Show that $f = h^\lambda$, for some constant λ, and deduce that

$$h^\lambda \frac{\mathrm{d}h}{\mathrm{d}x} = \text{constant}.$$

Infer that either $U(x) \propto (x - x_0)^{(\lambda-1)/(\lambda+1)}$, for $\lambda \neq -1$, or $U(x) \propto \mathrm{e}^{cx}$, where x_0 and c are constants. Deduce that the boundary layer thickness is only constant when $U(x)$ is linear in x. To what physical situation does this correspond?

36 ·a 2004

3. Suppose that a two-dimensional jet emanates in the x-direction from a nozzle at the origin into quiescent fluid. Starting from the steady dimensionless Navier-Stokes equations

$$(\mathbf{u}\cdot\nabla)\mathbf{u} = -\nabla p + \frac{1}{\mathrm{Re}}\nabla^2\mathbf{u}, \qquad \nabla\cdot\mathbf{u} = 0,$$

show that in $x > 0$ the jet is described by

$$\begin{aligned} u\frac{\partial u}{\partial x} + V\frac{\partial u}{\partial Y} &= -\frac{\partial p}{\partial x} + \frac{\partial^2 u}{\partial Y^2}, \\ 0 &= \frac{\partial p}{\partial y}, \\ \frac{\partial u}{\partial x} + \frac{\partial V}{\partial Y} &= 0, \end{aligned}$$

after a suitable rescaling. What are the appropriate boundary conditions on $Y = 0$ and as $Y \to \infty$? Show that the boundary layer equations imply

$$\int_{-\infty}^{\infty} u^2\,\mathrm{d}Y = \text{ constant.}$$

What is the physical interpretation of this result? What is the physical interpretation of the fact that $\int_{-\infty}^{\infty} u\,\mathrm{d}Y$ is not constant?

Show that there is a solution of the form $\Psi = x^{1/3} f(\eta)$ where $\eta = Y/x^{2/3}$ and Ψ is the streamfunction. What are the equation and boundary conditions satisfied by f?

B6.a. 2005.

2. Prandtl's boundary layer equations are

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \nu\frac{\partial^2 u}{\partial y^2} + U\frac{\mathrm{d}U}{\mathrm{d}x}, \qquad \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0. \tag{1}$$

Define all the variables in these equations and state the conditions under which these equations hold.

Show that the complex potential $w = A \log z$ represents an inviscid flow in a wedge of angle α with a source of strength $A\alpha$ at the origin. Explain why, in high Reynolds number flow, boundary layers will develop on the fixed boundaries.

The boundaries of the flow are at $y = 0$ and $y = x \tan \alpha$. Show that there is a boundary layer on $y = 0$ where equations (1) hold with $U = Ax^{-1}$. What are the boundary conditions in this case? Show that there is a solution of the form $\psi = f(y/x)$ where ψ is the stream function and f satisfies

$$\nu f''' + (f')^2 = A^2$$

with $f(0) = f'(0) = 0$ and $f'(\eta) \to A$ as $\eta \to \infty$.

Show that this problem has no solution if $A > 0$. What are the implications of this result?

B6.a 2006

A. Viscous Flow

1. The non-dimensional form of the Navier-Stokes equation for steady flow is

$$(\mathbf{u}.\boldsymbol{\nabla})\mathbf{u} = -\boldsymbol{\nabla}p + \frac{1}{Re}\nabla^2\mathbf{u},$$

$$\boldsymbol{\nabla}.\mathbf{u} = 0.$$

When the Reynolds number Re is large, derive Prandtl's boundary layer equations for a flow $\mathbf{u} = U(x)\mathrm{i}$ past a flat plate at $y = 0$, $x > 0$, in the form

$$u\frac{\partial u}{\partial x} + V\frac{\partial u}{\partial Y} = U\frac{\mathrm{d}U}{\mathrm{d}x} + \frac{\partial^2 u}{\partial Y^2}, \qquad \frac{\partial u}{\partial x} + \frac{\partial V}{\partial Y} = 0,$$

where the variables Y, V should be defined.

When $U(x) = \mathrm{e}^{\alpha x}$, show that there is a similarity solution for the stream function ψ of the form

$$\psi = \mathrm{e}^{\beta x} f\left(Y\mathrm{e}^{\gamma x}\right),$$

where β, γ should be determined, and f satisfies

$$\alpha f'^2 - \frac{\alpha}{2} f f'' = \alpha + f'''.$$

What are the boundary conditions for f?

Deduce that the width of the boundary layer will increase/decrease as x increases according as the external flow is decelerating/accelerating in x. Is this observation true in more general flows? Is this boundary layer approximation likely to be an equally good approximation for all values of α?

B.6a 2007

2. A two-dimensional jet of viscous fluid of known momentum flux issues in the x-direction through a narrow nozzle at the origin into an infinite ocean of the same fluid which is at rest at infinity. Starting from the non-dimensional Navier–Stokes equations for steady flow

$$(\mathbf{u}.\nabla)\mathbf{u} = -\nabla p + \frac{1}{Re}\nabla^2\mathbf{u}, \qquad \nabla.\mathbf{u} = 0,$$

where $\mathbf{u} = \big(u(x,y), v(x,y)\big)$, and assuming that the boundary layer approximation will be valid in the jet sufficiently far downstream, derive the boundary layer equations

$$u\frac{\partial u}{\partial x} + V\frac{\partial u}{\partial Y} = \frac{\partial^2 u}{\partial Y^2}, \qquad \frac{\partial u}{\partial x} + \frac{\partial V}{\partial Y} = 0,$$

where V and Y are scaled variables which should be defined.

What are the appropriate boundary conditions at $Y = 0$ and as $Y \to \infty$?

Show that $\int_{-\infty}^{\infty} u^2 \mathrm{d}Y$ is constant and explain the physical significance of this result.

Show that there is a similarity solution of the form

$$u = x^{\alpha} f'(Y/x^{\beta}),$$

where $\alpha = -\frac{1}{3}$, $\beta = \frac{2}{3}$ and f satisfies the third order ordinary differential equation

$$3f''' + ff'' + (f')^2 = 0,$$

with $f(0) = f''(0) = 0$ and $f'(\infty) = 0$. Hence show that

$$u = 6A^2 x^{-1/3}\mathrm{sech}^2(AYx^{-2/3}),$$

and indicate how the constant A can be determined.

B.6a 2008.

2. (i) The non-dimensional form of the Navier-Stokes equation for steady flow is

$$(\mathbf{u}.\nabla)\mathbf{u} = -\nabla p + \frac{1}{Re}\nabla^2\mathbf{u},$$

$$\nabla.\mathbf{u} = 0.$$

When the Reynolds number Re is large, derive Prandtl's boundary layer equations for a flow $\mathbf{u} = U(x)\mathbf{i}$ past a flat plate at $y = 0$, $x > 0$, in the form

$$u\frac{\partial u}{\partial x} + V\frac{\partial u}{\partial Y} = U\frac{\mathrm{d}U}{\mathrm{d}x} + \frac{\partial^2 u}{\partial Y^2}, \qquad \frac{\partial u}{\partial x} + \frac{\partial V}{\partial Y} = 0,$$

where the variables Y, V should be defined. State the boundary conditions on u, V. Comment briefly on what happens near $(0, 0)$.

(ii) When $U(x) = x^m$, show that there is a similarity solution of these boundary layer equations for the stream function ψ of the form

$$\psi = x^\beta f(Y/x^\gamma),$$

where β, γ should be determined, and f satisfies

$$f''' + \beta f f'' + m(1 - f'^2) = 0.$$

What are the boundary conditions for f?

By defining $w = f'$, find a second order differential equation for $w(f)$. What boundary conditions does $w(f)$ satisfy?

B6.a. 2009

2. The dimensionless two-dimensional steady Navier–Stokes equations are given by

$$\nabla \cdot \mathbf{u} = 0, \quad (\mathbf{u} \cdot \nabla)\mathbf{u} = -\nabla p + \frac{1}{Re}\nabla^2 \mathbf{u},$$

where $\mathbf{u}(x, y, t) = u\mathbf{i} + v\mathbf{j}$ is the velocity, $p(x, y, t)$ is the pressure and Re is the Reynolds number. A uniform stream of fluid with unit upstream velocity in the x-direction flows past a semi-infinite flat plate at $y = 0$, $x > 0$.

(a) When the Reynolds number Re is large, show that at leading order the flow in a boundary layer of thickness of $O\left(Re^{-1/2}\right)$ on the top of the plate is governed by Prandtl's boundary layer equations

$$\frac{\partial u}{\partial x} + \frac{\partial V}{\partial Y} = 0, \quad u\frac{\partial u}{\partial x} + V\frac{\partial u}{\partial Y} = -\frac{\partial p}{\partial x} + \frac{\partial^2 u}{\partial Y^2}, \quad 0 = -\frac{\partial p}{\partial Y},$$

where the variables Y and V should be defined.

State the boundary conditions on u and V, and deduce that $\partial p/\partial x = 0$.

(b) Denote by $\Psi(x, Y)$ the stream function that satisfies

$$u = \frac{\partial \Psi}{\partial Y}, \quad V = -\frac{\partial \Psi}{\partial x},$$

and assume that Ψ is zero on the plate and strictly increasing with $Y > 0$. By changing dependent variables from (x, Y) to (x, Ψ), writing $u(x, Y) = U(x, \Psi)$ and using the chain rule, show that

$$\frac{\partial u}{\partial x} = \frac{\partial U}{\partial x} - V\frac{\partial U}{\partial \Psi},$$

and obtain similar expressions for $\partial u/\partial Y$ and $\partial^2 u/\partial Y^2$. Hence show that the boundary layer equations imply

$$\frac{\partial U}{\partial x} = \frac{\partial}{\partial \Psi}\left(U\frac{\partial U}{\partial \Psi}\right).$$

Show that a solution exists of the form $U = g(\eta)$, $\eta = \Psi/\sqrt{x}$, where

$$\frac{\mathrm{d}}{\mathrm{d}\eta}\left(g\frac{\mathrm{d}g}{\mathrm{d}\eta}\right) + \frac{1}{2}\eta\frac{\mathrm{d}g}{\mathrm{d}\eta} = 0.$$

What boundary conditions does $g(\eta)$ satisfy?