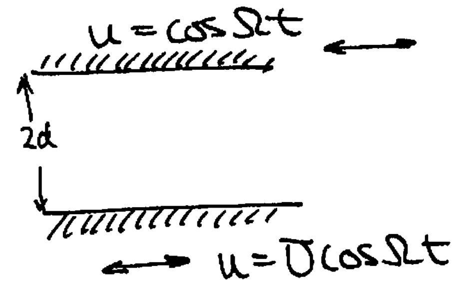
2009

#1 First parts are standard.

(b)



Assuming $\underline{u} = [u(y,t), 0]$, clearly the NS eqns reduce to:

$$\rho u_t = \mu u_{yy}$$

Now let $u = \mathcal{R}e[Uf(y)e^{i\Omega t}]$ '=' $Uf(y)e^{i\Omega t}$

(ignore real parts since problem is linear)

B.C.s $u(\pm d, t) = U\cos\Omega t \Rightarrow$ need $f(\pm d) = 1$

Substituting gives:

$$f'' - \frac{\rho i\Omega}{\mu} f = 0$$

Let $f = e^{ry} \Rightarrow r^2 = \frac{\rho\Omega}{\mu} \cdot i$

$$\Rightarrow r = \pm\underbrace{\sqrt{\frac{\rho\Omega}{2\mu}}}_{k}(1+i)$$

$$\therefore f(y) = A\cosh k(1+i)y + B\sinh k(1+i)y$$

(much easier form for B.C.s)

$$f(\pm d) = 0 \Rightarrow B = 0 \;\; ; \;\; A = \frac{1}{\cosh k(1+i)d}$$

As $kd \to \infty$, $f(y) = \frac{\cosh k(1+i)y}{\cosh k(1+i)d} \sim \frac{2}{e^{kd(1+i)}} \cdot \cosh k(1+i)dy$

exp. large.

So u is exp. small, except if $y \sim O(\frac{1}{k})$

2008

#1. a) Prove the transport theorem.

$$\frac{d}{dt}\iiint_{V(t)} F(\underline{x},t)\cdot dV = \frac{d}{dt}\iiint_{V(0)} F\cdot J\cdot dX_1\, dX_2\, dX_3$$

$$= \iiint_{V(0)} \frac{D}{Dt}(FJ)\cdot dX_1 dX_2\, dX_3$$

$$= \iiint_{V(0)} \frac{DF}{Dt}J + F\cdot\frac{DJ}{Dt}\quad dX_1\, dX_2\, dX_3$$

$$= \iiint_{V(t)} \left(\frac{DF}{Dt} + F\frac{DJ}{Dt}\frac{1}{J}\right) dV$$

$$= \iiint_{V(t)} \frac{DF}{Dt} + F(\nabla\cdot u)\cdot dV$$

$$= \iiint_{V(t)} \frac{\partial F}{\partial t} + \nabla\cdot(Fu)\quad dV \qquad \square$$

now by Euler's Thm,

$$\frac{DJ}{Dt} = J(\nabla\cdot u)$$

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + (u\cdot\nabla)$$

Prove Euler's Thm: Write $J = e_{ijk}\frac{\partial x_1}{\partial X_i}\frac{\partial x_2}{\partial X_j}\frac{\partial x_3}{\partial X_k}$

$$\frac{DJ}{Dt} = e_{ijk}\left\{\frac{\partial u_1}{\partial X_i}\frac{\partial x_2}{\partial X_j}\frac{\partial x_3}{\partial X_k} + \frac{\partial x_1}{\partial X_i}\frac{\partial u_2}{\partial X_j}\frac{\partial x_3}{\partial X_k} + \frac{\partial x_1}{\partial X_i}\frac{\partial x_2}{\partial X_j}\frac{\partial u_3}{\partial X_k}\right\}$$

(chain rule) $$= e_{ijk}\left\{\frac{\partial u_1}{\partial x_m}\frac{\partial x_m}{\partial X_i}\frac{\partial x_2}{\partial X_j}\frac{\partial x_3}{\partial X_k} + \frac{\partial u_2}{\partial x_m}\frac{\partial x_1}{\partial X_i}\frac{\partial x_m}{\partial X_j}\frac{\partial x_3}{\partial X_k} + \frac{\partial u_3}{\partial x_m}\frac{\partial x_1}{\partial X_i}\frac{\partial x_2}{\partial X_j}\frac{\partial x_m}{\partial X_k}\right\}$$

$$= \frac{\partial u_1}{\partial x_m}\frac{\partial(x_m,x_2,x_3)}{\partial(X_i,X_j,X_k)} + \frac{\partial u_2}{\partial x_m}\frac{\partial(x_1,x_m,x_3)}{\partial(X_i,X_j,X_k)} + \frac{\partial u_3}{\partial x_m}\frac{\partial(x_1,x_2,x_m)}{\partial(X_i,X_j,X_k)}$$

$$= \left(\frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} + \frac{\partial u_3}{\partial x_3}\right)J$$

$$= (\nabla\cdot u)J$$

b) Consv. of mass:

$$\frac{D}{Dt}\iiint_{V(t)} \rho \cdot dV = \iiint_{V(t)} \frac{\partial \rho}{\partial t} + \nabla\cdot(\rho u)\cdot dV = 0$$

Incompressible $\Rightarrow \frac{D\rho}{Dt} = \frac{\partial \rho}{\partial t} + (u\cdot\nabla)\rho = 0$

$$\therefore \underbrace{\frac{\partial \rho}{\partial t} + (u\cdot\nabla)\rho}_{=0} + \rho(\nabla\cdot u) = 0 \Rightarrow \boxed{\nabla\cdot u = 0}$$

Consev. of momentum:

$$\frac{D}{Dt}\iiint_{V(t)} \rho u_i \, dV = \iiint_{V(t)} \rho g_i \, dV + \iint_{\partial V} \sigma_{ij} n_j \cdot dS.$$

$$\Rightarrow \frac{\partial(\rho u_i)}{\partial t} + \nabla\cdot(\rho u_i \underline{u}) = \rho g_i + \frac{\partial \sigma_{ij}}{\partial x_j}$$

But $\frac{\partial \sigma_{ij}}{\partial x_j} = \frac{\partial}{\partial x_j}\left\{-p\delta_{ij} + \mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)\right\} = -\frac{\partial p}{\partial x_i} + \mu\frac{\partial}{\partial x_j}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$

$=0$ since $\nabla\cdot\underline{u} = 0$.

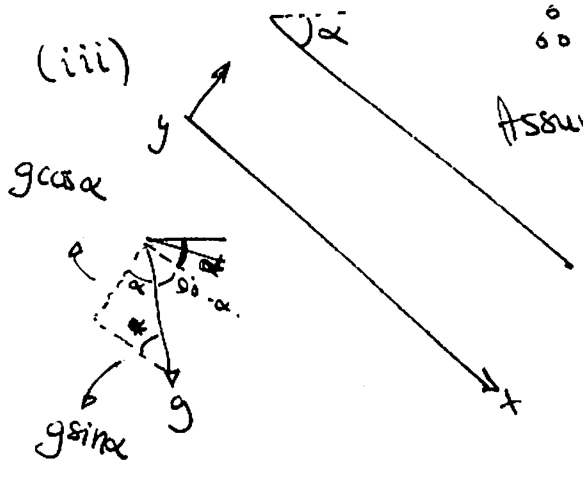
$$\therefore \cancel{\frac{\partial \rho}{\partial t} u_i + u_i(\nabla\cdot\rho) + \rho\left(\frac{\partial u_i}{\partial t} + \nabla u_i\right)}$$

note LHS $= \frac{\partial \rho}{\partial t} u_i + \frac{\partial u_i}{\partial t}\rho + \nabla u_i\cdot(\rho \underline{u}) + u_i \nabla\cdot(\rho\underline{u})$

$$= u_i\underbrace{\left\{\frac{\partial \rho}{\partial t} + \nabla\cdot(\rho\underline{u})\right\}}_{=0.} + \rho\left(\underbrace{\frac{\partial u_i}{\partial t} + \underline{u}\cdot\nabla u_i}_{\frac{Du}{Dt}}\right)$$

$$\therefore \rho\frac{Du}{Dt} = -\nabla p + \rho g + \mu\nabla^2\underline{u}$$

(iii)



$\therefore \underline{g} = [g\sin\alpha, g\cos\alpha]$.

Assume uni-directional $\Rightarrow \underline{u} = [u(y), 0]$

$$\therefore \cancel{u_t} + \cancel{u u_x} + \cancel{v u_y} = -p_x + (\cancel{u_{xx}} + u_{yy})\mu + \rho g\sin\alpha$$

$$\cancel{v_t} + \cancel{u v_x} + \cancel{v v_y} = -p_y + \mu(\cancel{v_{xx}} + \cancel{v_{yy}}) + \rho g\cos\alpha$$

$\therefore p_y = \rho g\cos\alpha$

$\Rightarrow p = (\rho g\cos\alpha)y + f(x)$

$\therefore p_x = f'(x)$ and $p_x = \mu u_{yy} + \rho g \sin\alpha$

$$\Rightarrow \underbrace{\mu u_{yy}}_{\text{fn. of } \cancel{x}y} = \underbrace{p_x - \rho g \sin\alpha}_{\text{fn. of } x}$$

$\therefore \mu u_{yy} = -G - g\sin\alpha = -(\rho g \sin\alpha + G)$ where

$$G = -p_x = \underline{\text{const}}.$$

<u>B.C.s?</u> Bottom plate still $\Rightarrow u(0) = 0$.

Top plate moves with fluid.

Stress on top $\Rightarrow F_i = \sigma_{ij} n_j = \sigma_{i2} n_2$

$$\sigma_{i2} = \mu\left(\frac{\partial u_i}{\partial x_2} + \cancel{\frac{\partial u_2}{\partial x_i}}\right) - p\delta_{i2}$$

$\hat{n} = [0, -1]$ only element is $\sigma_{12} = \mu \dfrac{\partial u}{\partial y}(h)$

$\therefore$ shear stress is $-\mu \dfrac{\partial u}{\partial y}(h) = \tau$ (why take normal <u>into</u> fluid?)

Then $\mu u_{yy} = -(\rho g \sin\alpha + G)$

$\Rightarrow \mu u_y = -(\rho g \sin\alpha + G) y + \text{const}.$

@ $y = h \Rightarrow \tau = -(\rho g \sin\alpha + G) h + C$

$\therefore \mu u_y = -(\rho g \sin\alpha + G) y + \tau + (g \sin\alpha + G) h$

$\mu u_y = (\rho g \sin\alpha + G)(h - y) + \tau$

$\Rightarrow \mu u = (\rho g \sin\alpha + G)(hy - \tfrac{1}{2}y^2) + \tau y + D$

$u(0) = 0 \Rightarrow D = 0$

Then using $u = U$ on $y = h$

$\Rightarrow \mu U = \tau h + (\rho g \sin\alpha + G) h^2 (1 - \tfrac{1}{2})$

Then...

$$u = \underbrace{\frac{1}{\mu}\left\{\frac{\tau}{h} + G + \rho g \sin\alpha\right\}}_{\beta\gamma g \sin\alpha} hy - \underbrace{\frac{1}{2\mu}(G + \rho \sin\alpha)}_{\beta g \sin\alpha} y^2$$

$$= \beta y(\gamma - y)\, g \sin\alpha.$$

Moreover, the shear/unit length on lower plate is.

$$\sigma_{12}\Big|_{y=0} = \mu \cdot \frac{\partial u}{\partial y}(0)$$

$$= (\rho g \sin\alpha + G)\, h + \tau$$

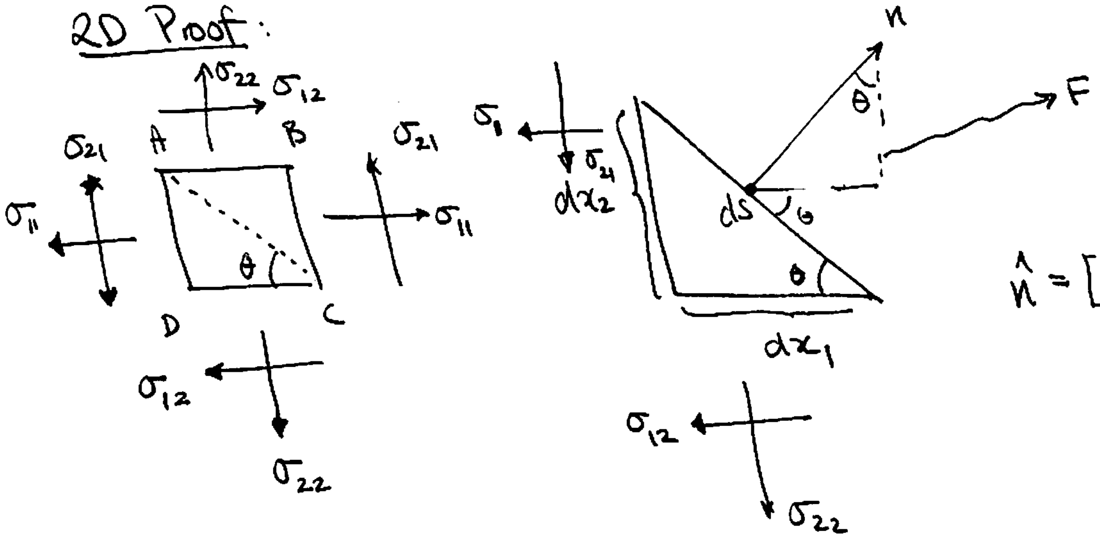
2007

#1 Remember: Stress vector τ : Force per area on surface element with normal $\underline{n}$ by fluid.

Stress tensor σ_{ij} : Force per unit area in i^{th} direction on surface with normal e_j.

Cauchy's Stress Thm.: $\tau_i(\hat{n}) = \sigma_{ij} n_j$

2D Proof:



$\hat{n} = [\sin\theta, \cos\theta]$.

Consider force on point with element dS. By force balance.

$$dS \cdot \tau_1 = \sigma_{12} dx_1 + \sigma_{11} \cdot dx_2$$

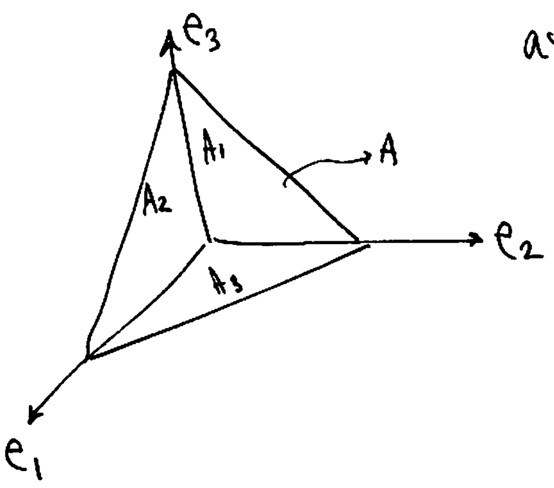
$$\Rightarrow \quad \tau_1 = \sigma_{12} \frac{dx_1}{dS} + \sigma_{11} \frac{dx_2}{dS}$$

$$= \sigma_{12} \cdot \underset{n_2}{\underset{\uparrow}{\cos\theta}} + \sigma_{11} \cdot \underset{n_1}{\underset{\uparrow}{\sin\theta}}$$

$\tau_1 = \sigma_{ij} n_j$ and similarly for τ_2

(Technically, we should have used θ_1, θ_2 and $\theta_1 \neq \theta_2$)

∴ Essentially the same in 3-D:



assume area of $A \sim L^2$.

By Newton's Law:

$$\underbrace{\iiint_V \rho \cdot \frac{Du}{Dt} - \rho F \, dV}_{O(L^3) \text{ by MVT.}} = \iint_{\partial V} \tau \cdot dS.$$

∴ We have force balance over the faces:

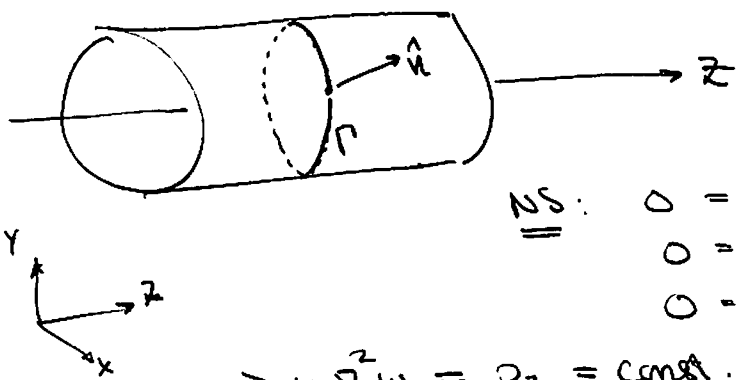
$$\tau_i(\hat{n}) \cdot L^2 + \tau(-e_j) \cdot L^2 \cdot n_j = O(L^3) \quad \text{as } L \to 0.$$

$$\Rightarrow \tau_i(\hat{n}) L^2 = \tau(e_j) n_j \cdot L^2 \quad \text{as } L \to 0$$

$$\therefore \text{ need } \tau_i(\hat{n}) = \underbrace{\tau(e_j)}_{\sigma_{ij}} n_j = \sigma_{ij} n_j$$

$\boxtimes$

Proof of momentum + NS eqns are the same.



Axi-symmetric:
$\underline{u} = [0, 0, w(x,y)]$.

$\underline{\underline{NS}}$:
$$0 = -p_x$$
$$0 = -p_y$$
$$0 = -p_z + \mu(w_{xx} + w_{yy})$$

$$\Rightarrow \mu \nabla^2 w = p_z = \text{const}.$$

$$\text{Drag} = -\int_\Gamma \tau_3 \cdot dS = -\int_\Gamma \sigma_{31} n_1 + \sigma_{32} n_2 \cdot dS.$$

(assuming outward normal from exterior)

note negative sign is force from wall on fluid. It would be positive force force on wall from fluid.

note $\sigma_{31} = \mu w_x$, $\sigma_{32} = \mu w_y$

$$\Rightarrow D = -\int_\Gamma \mu \nabla w \cdot \hat{n} \cdot dS = -\iint_V \mu \nabla^2 w \cdot dV = -p_z \cdot \underline{\underline{A}}$$

2006

#4 Deriving the energy eqn:

Assuming c_v & k are constant...

$$\sigma_{ij} n_j u_i = (\sigma_{ij} u_i) n_j$$

$$\underbrace{\frac{d}{dt}\iiint_{V(t)} \rho c_v T + \tfrac{1}{2} u_i^2 \cdot dV}_{\text{change in internal n.r.g.}} = \underbrace{\iint_{\partial V} q \cdot (-n) \cdot dS}_{\text{r.o. heat flux into } V(t)} + \underbrace{\iint_{\partial V} \tau \cdot u \cdot dS}_{\text{r.o. work done by surface stresses on } V(t)} + \underbrace{\iiint_{V(t)} \rho F \cdot \underline{u} \cdot dV}_{\text{body forces}}$$

By Fourier's Law:

$$\iint_{\partial V} -k\nabla T \cdot (-\underline{n}) \cdot dS$$

$$\therefore \frac{d}{dt}\iiint_{V(t)} \rho c_v T + \frac{\rho}{2} u_i^2 \cdot dV = \iiint_{V(t)} \frac{\partial}{\partial x_j}\left(k\frac{\partial T}{\partial x_j}\right) + \frac{\partial}{\partial x_j}(\sigma_{ij} u_i) + \rho F_i u_i \cdot dV$$

note $\displaystyle \frac{d}{dt}\iiint_{V(t)} (\rho f) \cdot dV = \rho \iiint_{V(t)} \frac{Df}{Dt} \cdot dV$ if incompressible.

$$\therefore \rho c_v \cdot \frac{DT}{Dt} + u_i\left(\rho \frac{Du_i}{Dt} - \frac{\partial}{\partial x_j}\sigma_{ij} - \rho F_i\right) = k\nabla^2 T + \underbrace{\sigma_{ij}\frac{\partial u_i}{\partial x_j}}_{\Phi}$$

oops x

$$\therefore \rho c_v \cdot \frac{DT}{Dt} = k\nabla^2 T + \Phi.$$

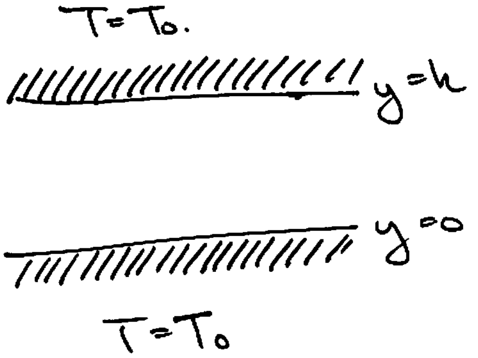
Note $\displaystyle \Phi = \sigma_{ij}\frac{\partial u_i}{\partial x_j} = \left(-p\delta_{ij} + \mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)\right)\frac{\partial u_i}{\partial x_j}$

write $\displaystyle e_{ij} = \frac{\mu}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) \Rightarrow \Phi = 2\mu e_{ij}\frac{\partial u_i}{\partial x_j}$

Then $\Phi = \mu e_{ij} \frac{\partial u_i}{\partial x_j} + \mu e_{ji} \frac{\partial u_j}{\partial x_i}$

$= \mu e_{ij} \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right)$ using $e_{ij} = e_{ji}$

$= \frac{\mu}{2} \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right)^2$



$\underline{u} = \left[\frac{Uy}{h}, 0 \right]$.

assume $T = T(y)$

Then $\rho c_v \left(\frac{\partial T}{\partial t} + u_i \frac{\partial T}{\partial x_i} \right) = k \nabla^2 T + \frac{\mu}{2} \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right)^2$

$\Rightarrow \rho c_v (T_t) = k T_{yy} + \frac{\mu}{2} \left(\frac{\partial u}{\partial y} \right)^2$

$\rho c_v T_t = k T_{yy} + \frac{\mu U^2}{h^2}$

Steady-state:

$T_{yy} = -\frac{\mu U^2}{k h^2} \Rightarrow T = T_0 + \frac{\mu U^2}{k h^2} (h-y) y$.

Time dependent? Write $\boxed{T = \overline{T} + T_1 + \frac{\mu U^2}{k h^2} y(h-y)}$ with $T(y,t) = T_1$ @ $y = 0, h$

$\Rightarrow \overline{T}(0,h,t) = 0$

Then:

$\rho c_v \overline{T}_t = k \overline{T}_{yy}$

$\Rightarrow$ write $\overline{T} = f(t) g(y) \Rightarrow \rho c_v f' g = k f g''$

$\Rightarrow \frac{f'}{f} = \underbrace{\frac{k}{\rho c_v}}_{=\frac{1}{A}} \frac{g''}{g} = -\lambda$

Clean-up:

Write $T(y,t) = \overline{T} + T_1 + \frac{\mu U^2}{kh^2} y(h-y)$

$T(y,0) = T_s \Rightarrow \overline{T}(y,0) = T_0 - T_1$

$\left.\begin{array}{l} T(0,t) = T_1 \\ T(h,t) = T_1 \end{array}\right\} \Rightarrow \overline{T}(0,t) = 0 = \overline{T}(h,t)$

$\therefore \rho c_v \overline{T}_t = k \overline{T}_{yy}$

Let $\overline{T} = f(t)g(y) \Rightarrow \frac{f'}{f}\frac{\rho c_v}{k} = \frac{g''}{g} = -\lambda^2$

$\therefore f = Ce^{-\frac{\lambda^2 \rho c_v}{k} t}$

Also: $g'' = -\lambda^2 g \Rightarrow g = D\cos\lambda y + E\sin\lambda y$

$g(0) = 0 \Rightarrow D = 0$

$g(h) = 0 \Rightarrow \lambda h = n\pi \rightarrow \lambda = \frac{n\pi}{h}$

$\therefore \overline{T} = \sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi}{h} y\right) e^{-\frac{n^2\pi^2}{h^2}\cdot\frac{\rho c_v}{k}\cdot t}$

Apply $\overline{T}(y,0) = T_0 - T_1 \Rightarrow T_0 - T_1 = \sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi y}{h}\right)$

$\therefore a_n = \frac{2}{h}\int_0^h \sin\left(\frac{n\pi y}{h}\right)(T_0 - T_1)\,dy$

$= \frac{2(T_0 - T_1)}{n\pi}\left[(-1)^n + 1\right]$

$\therefore a_{2n} = 0$; $a_{2n+1} = \frac{4(T_0 - T_1)}{(2n+1)\pi}$

$$\therefore \quad T(y,t) = T_1 + \frac{\mu U^2}{k h^2} y(h-y) + \sum_{n=1}^{\infty} \frac{4(T_0 - T_1)}{(2n+1)\pi} \sin\left[\frac{(2n+1)\pi}{h} y\right]$$
$$\times e^{\frac{-k}{\rho c h^2}(2n+1)^2 \pi^2 t} .$$

If the boundary is insulated @ $t=0$

$$\Rightarrow T_y(0,0) = 0 \quad \& \quad T_y(h,0) = 0 .$$

$$k \cdot T_{yy} = \frac{\mu U^2}{h^2} \Rightarrow T_y = \frac{\mu U^2}{k h^2} y + \underset{=0}{C}$$

no way to satisfy both $\therefore$ no steady sol'n.

To solve with ~~these~~ conditions, seek "steady sol'n" in $T = T(t)$

$$\therefore \rho c_v T_t = \frac{\mu U^2}{h^2} \Rightarrow T = C e^{\frac{\mu U^2}{h^2 \rho c_v} \cdot t}$$
$$\Rightarrow T_y \equiv 0 \text{ for arbitrary } C.$$

WLOG, set $C=1$ and linearise around this sol'n.

Let $T = e^{\frac{\mu U^2}{h^2 \rho c_v} t} + \overline{T} \Rightarrow \rho c_v \cdot \overline{T}_t = k \overline{T}_{yy}$.

now separate and solve subject to $\overline{T}_y = 0$ on $y = 0, h$.

$$T = T_1 + \frac{\mu U^2}{2kh^2} y(h-y) + \Theta$$

$\Theta = 0 \quad y = 0, h,$

$\Theta = T_0 - T_1 \quad$ on $\quad t = 0$

$$\Theta = \sum_1 a_n \sin\left(\frac{n\pi y}{h}\right) e^{\frac{-kn^2\pi^2 t}{\rho c h^2}}$$

$$T_0 - T_1 = \sum_1^\infty \sin\left(\frac{n\pi y}{h}\right) \cdot a_n \quad \text{in} \quad 0 < y < h$$

$$\Rightarrow a_n = \frac{2}{h}\int_0^h (T_0 - T_1) \sin\frac{n\pi y}{h}\, dy$$

$$= \frac{2(T_0 - T_1)}{n\pi}\left\{(-1)^{n+1} + 1\right\}$$

$$a_{2n} = 0 \qquad a_{2n+1} = \frac{4(T_0 - T_1)}{(2n+1)\pi}$$

Insulated walls : $T_y = 0$ on $y = 0, h$

no sol'n to steady state.

But $T = \frac{\mu U^2}{e c k^2} t$ satisfies eqn for T.

∴ Write $T = \frac{\mu U^2}{e c k^2} \cdot t + \Theta$

$$\Rightarrow \rho c \Theta_t = k \Theta_{yy} \qquad \Theta_y = 0 \text{ on } y = 0, h$$

$$\Theta = T_0 + \frac{\mu U^2}{2kh^2} y(h-y) \text{ @ } t = 0$$

Separate to get

$$\Theta = \sum b_n \cos\left(\frac{n\pi}{h} y\right) e^{-\frac{kn^2\pi^2}{e c k^2} t}.$$

#1 Width of B.L. is.. $y \sim O(e^{-\alpha/(2x)})$.

increases if $\alpha < 0$ i.e. $U' < 0$. decelerating.
decreases if $\alpha > 0$ i.e $U' > 0$. accelerating.

This trend is true for general flows.
$\Rightarrow$ acc. flows make B.L. thin.
dec. flows make B.L. thick

but switch (increasing to decreasing) doesn't always happen at switch from dec. to acc.

In particular, unif. flow has $y \sim x^{1/2}$ and
$U = x^{\alpha} \Rightarrow$ switch from growth to decay occurs
@ $\alpha = 1$

B.L. remains attached if $p' > 0$. Detach if $p' < 0$
and in this case $(\alpha < 0)$, B.L theory breaks down.

2005

#1 Same as 2003 except . . .

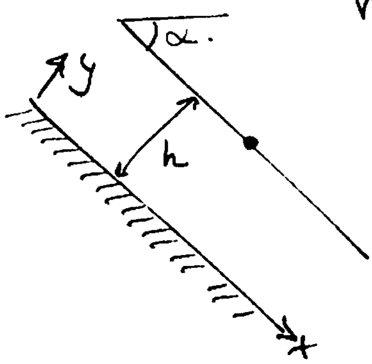
What are assumptions of Newtonian fluid?

(A) τ_{ij} is linear relation of velocity gradients

(B) τ_{ij} is invariant to rotations (isotropic)

e.g. stress tensor is symmetric.

Flow problem



has free surface:

need extra condition

(i) Kinematic

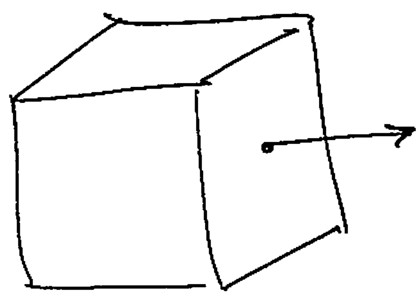
$$\frac{D}{Dt}(y-v) = 0$$

automatic.

(ii) Dynamic

$$\left.\frac{\partial u}{\partial y}\right|_{y=h} = 0.$$

$$t_i = \sigma_{ij} n_j$$



2004

#1. First part is standard.

$u=0$.

Assume that $\underline{u} = [u(y), 0]$.

Then clearly:

$u = U\cos\omega t$

$$\rho u_t = \mu u_{yy}$$

Write: $u = f(y)e^{i\omega t} \Rightarrow \rho i\omega \cdot f = \mu f''$

$$\Rightarrow r^2 = \frac{\rho i\omega}{\mu}$$

$$\Rightarrow r = \underbrace{\sqrt{\frac{\rho\omega}{2\mu}}}_{k}(1+i)$$

$$f(y) = \underbrace{Ae^{k(1+i)y}}_{\text{kill this mode}} + Be^{-k(1+i)y} \Rightarrow \text{need decay.}$$

$\therefore f(y) = Be^{-k(1+i)y}$. Need $f(0) = U \Rightarrow B = U$

Then $u(y) = Ue^{-k(1+i)y}e^{i\omega t}$

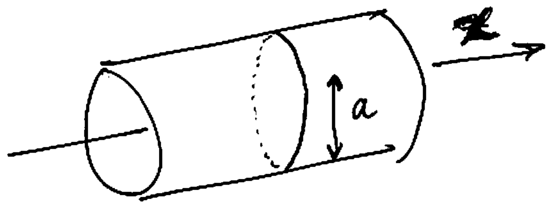
$$= Ue^{-ky}e^{i(\omega t - ky)}$$

$$\therefore \Re e(\ldots) = Ue^{-ky}\cos(\omega t - ky)$$

$$= Ue^{-\sqrt{\frac{\rho\omega}{2\mu}}\,y}\cos\left(\sqrt{\frac{\rho\omega}{2\mu}}\cdot y - \omega t\right)$$

2001

#1 First part is standard.



Let $\underline{u} = [u(\cancel{z}, y), 0, 0]$.

We have $\mu(u_{yy}+u_{zz}) = p_x = -G$

Transform to cylindrical coords.

$$\nabla^2 u = \frac{1}{r}\frac{\partial}{\partial r}\left(r\cdot\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\cdot\frac{\partial^2 u}{\partial \theta^2} + \frac{\partial^2 u}{\partial x^2}$$

Let $u(z,y) = u(r)$

$$\therefore \mu\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) = -G$$

$$\Rightarrow \frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) = -\frac{Gr}{\mu}$$

$$r\frac{\partial u}{\partial r} = -\frac{Gr^2}{2\mu} + C$$

$$\frac{\partial u}{\partial r} = -\frac{Gr}{2\mu} + \frac{C}{r}$$

bounded

$$\Rightarrow u = -\frac{Gr^2}{4\mu} + \cancel{C\cdot\log r} + D$$

$$u(a) = 0 \Rightarrow D = \frac{Ga^2}{4\mu} \Rightarrow u = \frac{G}{4\mu}(a^2 - r^2)$$

Flux: $$Q = \int_0^{2\pi}\int_0^a u\cdot r\cdot dr\cdot d\theta = 2\pi\int_0^a \frac{G}{4\mu}(a^2r - r^3)\cdot dr$$

$$= \frac{G\pi}{2\mu}\left(\frac{a^2}{2}a^2 - \frac{1}{4}a^4\right) = \frac{G\pi a^4}{8\mu}$$

2000

#4 $\rho u_t = \mu u_{yy}$.

$u = U\cos\omega t$ on $y=0$ $\quad u=0$ on $y=h$.

Let $u = f(y)e^{i\omega t}$ $\quad \alpha = (1+i)\sqrt{\dfrac{\rho\omega}{2\mu}}$

$$\left.\begin{aligned} \therefore\ f &= Ae^{\alpha y} + Be^{-\alpha y} \\ U &= A+B \\ 0 &= Ae^{\alpha h} + Be^{-\alpha h} \end{aligned}\right\} \Rightarrow A = \frac{-Ue^{-\alpha h}}{2\sinh\alpha h}$$

$$B = \frac{Ue^{\alpha h}}{2\sinh\alpha h}$$

$$u = \mathcal{R}e\left\{\frac{U\sinh\alpha(h-y)e^{i\omega t}}{\sinh\alpha h}\right\}$$

$$P = \text{force on plane} = \mu\frac{\partial u}{\partial y}\bigg|_{y=0} = \mathcal{R}e\left[-\mu U\alpha\coth\alpha h\, e^{i\omega t}\right]$$

$$h\to\infty \Rightarrow P \to \mathcal{R}e\left[-\mu U(1+i)\sqrt{\frac{\rho\omega}{2\mu}}(\cos\omega t + i\sin\omega t)\right].$$

$$P \sim U\sqrt{\rho\omega\mu}\,\frac{i}{\sqrt{2}}\left(-\cos\omega t + \sin\omega t\right)$$

$$= U\sqrt{\rho\omega\mu}\,\sin\left(\omega t - \pi/4\right).$$

1999

#1 $\frac{d}{dy}\left(\mu u_y\right) = 0 \quad kT_{yy} + \mu (u_y)^2 = 0.$

$\Rightarrow \mu u_y = C = \tau_w$ after having applied B.C.s.

$\therefore \quad kT_{yy} + \frac{\tau_w^2}{\mu} = 0$

no sol'n with $T_y = 0$ @ $y = 0, h$.

(iv): Let $\mu = \mu_0 e^{-T}$.

$\Rightarrow kT_{yy} + \frac{\tau_w^2}{\mu_0} e^{T} = 0.$ with say (?)

$T_* = 0$ on $y = 1$

$T_y = 0$ on $y = 0$

$T_0 = T(0)$

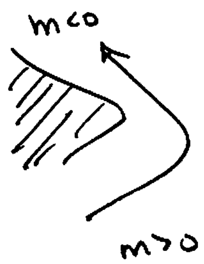
$\alpha = \frac{\tau_w^2}{k\mu_0} \quad \Rightarrow \frac{1}{2} T_y^2 + \alpha e^{T} = \alpha e^{T_0}.$

$\Rightarrow \int_0^{T_0} \frac{dT}{\sqrt{e^{T_0} - e^{T}}} = \sqrt{2\alpha}$

need $\alpha < \alpha_*$.

#2 Last part:

(iii) Trailing edge



Can't go to $m<0$

$\Rightarrow$ KJ (?)

#3

(iii) Solve by chars.

$$\frac{dt}{1} = \frac{dx}{\rho g h^2/\mu} = \frac{dh}{0}$$

$$h = F\left(x - \frac{\rho g h^2}{\mu} t\right), \quad F = h_0$$