B6.a. 2009

A. Viscous Flow

1. (a) State the Transport Theorem and Cauchy's Stress Theorem. The stress tensor in an incompressible Newtonian fluid of constant density ρ and constant viscosity μ is given by

$$\sigma_{ij} = -p\delta_{ij} + \mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right),$$

where p is the pressure, $\mathbf{u} = u_i\mathbf{e}_i$ is the velocity (using the summation convention) and $\mathbf{e}_i$ is the unit vector in the x_i-direction. By applying the principles of conservation of mass and momentum to an arbitrary material volume, derive the incompressible Navier–Stokes equations in the form

$$\nabla \cdot \mathbf{u} = 0, \quad \rho\left(\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u}\cdot\nabla)\mathbf{u}\right) = -\nabla p + \mu\nabla^2\mathbf{u}.$$

(b) Incompressible fluid occupies the region between two parallel rigid plates at $y = -d$ and $y = d$, both of which oscillate in the x-direction with speed $U\cos\Omega t$. Assuming that there is no applied pressure gradient and that the flow is unidirectional with speed $u(y,t)$ in the x-direction, show that

$$\rho\frac{\partial u}{\partial t} = \mu\frac{\partial^2 u}{\partial y^2}.$$

Deduce that u is given by the real part of $Uf(y)\mathrm{e}^{\mathrm{i}\Omega t}$, where

$$f(y) = \frac{\cosh k(1+\mathrm{i})y}{\cosh k(1+\mathrm{i})d}, \quad k = \sqrt{\frac{\rho\Omega}{2\mu}}.$$

What happens as $kd \to \infty$?

B6a. 2008

A. Viscous Flow

1. (i) Prove the transport theorem

$$\frac{\mathrm{d}}{\mathrm{d}t}\iiint\limits_{V(t)} F(\mathbf{x},t)\,\mathrm{d}V = \iiint\limits_{V(t)} \left(\frac{\partial F}{\partial t} + \nabla.(F\mathbf{u})\right)\mathrm{d}V,$$

where $V(t)$ is a material volume of fluid, F is a differentiable function defined at all points of the fluid, and $\mathbf{u}$ is the velocity of the fluid.

(ii) The stress tensor in a Newtonian fluid is given by

$$\sigma_{ij} = -p\,\delta_{ij} + \mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right),$$

where p is a scalar variable and μ is a constant parameter.

Assuming that in an incompressible fluid the velocity $\mathbf{u}$ satisfies the equation $\nabla.\mathbf{u} = 0$, show that it is possible to deduce the *Navier–Stokes Equations*

$$\rho\frac{\mathrm{D}\mathbf{u}}{\mathrm{D}t} = -\nabla p + \mu\nabla^2\mathbf{u} + \rho\mathbf{g},$$

where ρ is the density of the fluid and $\mathbf{g}$ is the acceleration due to gravity.

(iii) A layer of fluid of constant depth h flows steadily between two parallel planes inclined at an angle α to the horizontal. Choose the x-axis down the lower fixed plane and the y-axis perpendicular to the planes. Show that there is an appropriate steady solution of the Navier–Stokes equations in which the velocity between the planes is of the form

$$\beta y(\gamma - y)g\sin\alpha \quad \text{for} \quad 0 \leqslant y \leqslant h,$$

when the upper plane moves parallel to the lower plane such that it exerts shear force τ per unit length on the fluid. Here you should find β, γ to satisfy appropriate boundary conditions on the planes. What is the shear force per unit x length on the lower plane?

B6.a. 2007

A. Viscous Flow

1. Define the stress tensor σ_{ij} for a viscous fluid and show that the force per unit area exerted on a surface in the fluid with unit normal $\mathbf{n}$ is

$$\sigma_{ij} n_j \mathbf{e}_i,$$

where $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$ are unit vectors along the axes Ox_1, Ox_2, Ox_3.

By considering the momentum of a volume of fluid, show that

$$\rho \frac{\mathrm{d}u_i}{\mathrm{d}t} = \frac{\partial \sigma_{ij}}{\partial x_j},$$

where $\mathbf{u}$ is the velocity of the fluid, ρ is density and $\dfrac{\mathrm{d}}{\mathrm{d}t}$ is the convective derivative.

[*If you use the Transport Theorem to obtain this result you need not prove it but you should state it clearly.*]

Given that, for a Newtonian fluid,

$$\sigma_{ij} = -p\delta_{ij} + \mu \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right),$$

derive the Navier–Stokes equation for an incompressible viscous fluid

$$\rho \left(\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u}.\nabla)\mathbf{u} \right) = -\nabla p + \mu \nabla^2 \mathbf{u}.$$

Fluid flows steadily along a long straight cylindrical pipe of cross-sectional area A. Taking the z-axis along the pipe, show that the pressure gradient along the pipe is constant and that the velocity $w(x, y)\mathbf{k}$ satisfies

$$\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} = \frac{1}{\mu} \frac{\mathrm{d}p}{\mathrm{d}z}.$$

Show that the drag per unit length on the pipe is $-A\dfrac{\mathrm{d}p}{\mathrm{d}z}$.

B6.a. 2006

4. The equations for conservation of mass and momentum of an incompressible, viscous, conducting fluid can be written as

$$\frac{\partial u_i}{\partial x_i} = 0, \qquad \rho\frac{\mathrm{d}u_i}{\mathrm{d}t} = \frac{\partial \sigma_{ij}}{\partial x_j},$$

where the stress tensor $\sigma_{ij} = -p\delta_{ij} + \mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$.

Assuming these equations, derive the equation governing the temperature T of the fluid in the form

$$\rho c\left(\frac{\partial T}{\partial t} + u_i\frac{\partial T}{\partial x_i}\right) = k\frac{\partial^2 T}{\partial x_i^2} + \frac{\mu}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)^2.$$

[*If required, you may assume the Transport Theorem:*

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{V(t)} F\mathrm{d}V = \int_{V(t)}\left(\frac{\partial F}{\partial t} + \nabla.(F\mathbf{u})\right)\mathrm{d}V. \qquad]$$

The flow in a long two-dimensional channel $0 < y < h$ is given by $\mathbf{u} = \frac{Uy}{h}\mathbf{i}$, and the walls of the channel are held at constant temperature T_0. Show that a steady state solution is possible with

$$T = T_0 + \frac{\mu U^2}{2h^2 k}y(h-y).$$

The fluid is in this steady state when at $t = 0$ the temperature of the walls is suddenly changed to T_1, but the flow is unchanged. Show that for $t > 0$

$$T = T_1 + \frac{\mu U^2}{2h^2 k}y(h-y) + \sum_{n=1}^{\infty}\frac{4(T_0 - T_1)}{(2n+1)\pi}\sin\frac{(2n+1)\pi y}{h}\exp\left[-\frac{k}{\rho c h^2}(2n+1)^2\pi^2 t\right].$$

If, alternatively, the boundary is suddenly insulated at $t = 0$, show that there is no steady solution in this case, and explain (without detailed calculations) how the solution can be found for $t > 0$ in this case.

B6.a 2005

A. Viscous Flow

1. Prove the transport theorem

$$\frac{\mathrm{d}}{\mathrm{d}t}\iiint_{V(t)} F(\mathbf{x},t)\,\mathrm{d}V = \iiint_{V(t)} \left(\frac{\partial F}{\partial t} + \operatorname{div}(F\mathbf{u})\right)\mathrm{d}V$$

where $V(t)$ is a material volume of fluid, F is a differentiable function defined at all points of the fluid and $\mathbf{u}$ is the velocity of the fluid.

The stress tensor in a Newtonian fluid is given by

$$\sigma_{ij} = -p\,\delta_{ij} + \mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$$

where p is a scalar variable and μ is a constant parameter. What assumptions are made in deriving this relationship?

Assuming that in an incompressible fluid the velocity $\mathbf{u}$ satisfies the equation $\operatorname{div}\mathbf{u} = 0$, show that, by considering the rate of change of momentum of the fluid, it is possible to deduce the *Navier Stokes Equations*

$$\rho\frac{\mathrm{D}\mathbf{u}}{\mathrm{D}t} = -\nabla p + \mu\nabla^2\mathbf{u} + \rho\mathbf{g}$$

where ρ is the density of the fluid and $\mathbf{g}$ is the acceleration due to gravity.

A layer of fluid of constant depth h flows steadily down a plane inclined at an angle α to the horizontal. Choosing the x-axis down the plane and the y-axis perpendicular to the plane, show that there is an appropriate steady solution of the Navier Stokes equations in which the velocity down the plane is

$$\frac{\rho g y(2h-y)\sin\alpha}{2\mu}.$$

B6.a 2004

A. Viscous Flow

1. By considering conservation of mass in an arbitrary fluid volume, and by using the Transport theorem or otherwise, show that for an incompressible fluid

$$\nabla \cdot \mathbf{u} = 0.$$

Define the *stress tensor* σ_{ij}. By applying Newton's law to an arbitrary fluid volume, derive the conservation of momentum equation

$$\rho \frac{\mathrm{d}u_i}{\mathrm{d}t} = \frac{\partial \sigma_{ij}}{\partial x_j}.$$

Hence, by assuming that

$$\sigma_{ij} = -p\delta_{ij} + \mu \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right),$$

derive the Navier-Stokes equations for an incompressible, constant viscosity, Newtonian fluid, in the form

$$\rho \left(\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u} \right) = -\nabla p + \mu \nabla^2 \mathbf{u}.$$

Incompressible fluid occupies the region $y > 0$ above a plane rigid boundary $y = 0$ which oscillates to and fro in the x-direction with velocity $U \cos \omega t$. Show that $\mathbf{u} = (u(y,t), 0, 0)$ satisfies

$$\rho \frac{\partial u}{\partial t} = \mu \frac{\partial^2 u}{\partial y^2}$$

and by writing u as the real part of $f(y)e^{i\omega t}$ show that

$$u = U \exp\left(-\sqrt{\omega\rho/2\mu}\, y \right) \cos \left(\sqrt{\frac{\omega\rho}{2\mu}}\, y - \omega t \right).$$

What happens as $\omega \to \infty$?

B6.a 2003

A. Viscous flow

1. An incompressible fluid has dynamic viscosity $\mu(T)$, where T is the temperature. Assuming conservation of mass and momentum in the form (in the usual notation)

$$\sum_i \frac{\partial u_i}{\partial x_i} = 0, \quad \rho \frac{\mathrm{d}u_i}{\mathrm{d}t} = \sum_j \frac{\partial \sigma_{ij}}{\partial x_j},$$

where

$$\sigma_{ij} = -p\delta_{ij} + \mu \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right),$$

show that conservation of energy requires that

$$\rho c \frac{\mathrm{d}T}{\mathrm{d}t} = k \sum_i \frac{\partial^2 T}{\partial x_i^2} + \frac{\mu}{2} \sum_{i,j} \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right)^2,$$

where c is the specific heat and k is the thermal conductivity, which is assumed constant.

Suppose the fluid flows between a plate $y = 0$, which is stationary, and a parallel plate $y = h$, at which a shear stress of magnitude σ_w is applied. Show that there is a solution in which $x_2 = y$, $u_1 = u(y)$, $u_2 = u_3 = 0$, $T = T(y)$ and $p =$constant, provided

$$\frac{\mathrm{d}}{\mathrm{d}y} \left(\mu \frac{\mathrm{d}u}{\mathrm{d}y} \right) = 0, \quad k \frac{\mathrm{d}^2 T}{\mathrm{d}y^2} + \mu \left(\frac{\mathrm{d}u}{\mathrm{d}y} \right)^2 = 0.$$

Suppose further that the plates are held at the same constant temperature T_w and that $\mu = \mu_0 \mathrm{e}^{-T}$, where μ_0 is constant. Show that there exists a positive constant σ_w^* such that two solutions exist for $\sigma_w < \sigma_w^*$ and none exists for $\sigma_w > \sigma_w^*$.

[*You may assume that*

$$I(T_0) = \int_{T_w}^{T_0} \frac{\mathrm{d}T}{\sqrt{\mathrm{e}^{T_0} - \mathrm{e}^{T}}}, \quad T_0 \geqslant T_w,$$

increases from zero when $T_0 = T_w$ *to a maximum value and then tends to zero as* $T_0 \to \infty$.]

B6.a 2002

Viscous Flow

1. Fluid flows with velocity $\mathbf{u}(\mathbf{x}, t)$. Under what conditions on the boundary $\partial\Omega(t)$ of a domain $\Omega(t)$ does the transport theorem

$$\frac{\mathrm{d}}{\mathrm{d}t}\iiint_{\Omega(t)} F\,\mathrm{d}V = \iiint_{\Omega(t)}\left(\frac{\partial F}{\partial t} + \nabla\cdot(F\mathbf{u})\right)\mathrm{d}V \qquad (*)$$

hold for arbitrary smooth functions F?

Show that, if the stress tensor in a viscous fluid of density ρ is τ_{ij}, then

$$\frac{\mathrm{d}}{\mathrm{d}t}\iiint_{\Omega(t)} \rho u_i\,\mathrm{d}V = \iiint_{\Omega(t)} \frac{\partial \tau_{ij}}{\partial x_j}\,\mathrm{d}V, \text{ where } \mathbf{u} = (u_i), \mathbf{x} = (x_i).$$

Assuming the fluid is incompressible and Newtonian, so that

$$\tau_{ij} = -p\delta_{ij} + \mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$$

in the usual notation, use $(*)$ to deduce the Navier–Stokes equation

$$\rho\frac{\mathrm{d}\mathbf{u}}{\mathrm{d}t} = -\nabla p + \mu\nabla^2\mathbf{u}.$$

Show that, if the fluid flows unidirectionally down a cylindrical pipe aligned with the x-axis, where $\mathbf{x} = (x, y, z)$, then the x-velocity component u satisfies

$$\mu\left(\frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right) = C$$

where C is the pressure gradient. If the cross-section of the pipe is bounded by $y = 0$, $y \pm \sqrt{3}z = \sqrt{3}a$, show that

$$u = \frac{-C}{4\sqrt{3}a\mu}y\left((y - \sqrt{3}a)^2 - 3z^2\right).$$

B6.a 2001

Viscous Flow

1. Define what is meant by the *stress tensor* σ_{ij} and the *rate of strain tensor* $\dot{\varepsilon}_{ij}$.

   By consideration of torque balance for an infinitesimal fluid element, show that

$$\sigma_{ji} = \sigma_{ij}.$$

   By applying Newton's law to an arbitrary fluid volume, derive Cauchy's equation expressing conservation of momentum,

$$\rho\left(\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u}.\nabla)\,\mathbf{u}\right) = \mathbf{e}_i \frac{\partial \sigma_{ij}}{\partial x_j}.$$

   Hence, *assuming* that $\sigma_{ij} = -p\delta_{ij} + 2\mu\dot{\varepsilon}_{ij}$, derive the Navier-Stokes equations for an incompressible, constant viscosity, Newtonian fluid, in the form

$$\rho\left(\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u}.\nabla)\,\mathbf{u}\right) = -\nabla p + \mu\nabla^2\mathbf{u}, \quad \nabla.\mathbf{u} = 0.$$

   Fluid flows steadily down a cylindrical pipe of radius a under a constant pressure gradient $-G$. Show that the volume flux is

$$Q = \frac{\pi G a^4}{8\mu},$$

   where μ is the fluid viscosity.

4. What assumptions are needed to derive the relation

$$\sigma_{ij} = -p\delta_{ij} + \lambda\delta_{ij}\frac{\partial u_k}{\partial x_k} + \mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$$

for a Newtonian fluid?

Assuming this relation holds, derive the Navier–Stokes equations for an incompressible fluid with constant density ρ and viscosity μ. (You may assume the continuity equation $\nabla.\mathbf{u} = 0$ for an incompressible fluid.) Fluid is contained between the planes $y = 0$ and h and the plane at $y = 0$ oscillates so that its velocity in the x-direction is $U \cos \omega t$. Find the velocity in the fluid and show that the force per unit length on the plane $y = 0$ is given by the real part of $\{-\mu U \alpha e^{i\omega t} \coth \alpha h\}$ where $\alpha = (1 + i)\sqrt{\frac{\rho\omega}{2\mu}}$. Show that if h is large, the force oscillates with a frequency that is $\pi/4$ out of phase with the velocity and that the maximum value of the force is $U\sqrt{\omega\rho\mu}$.

[*You may leave your solution for the velocity field in the form of a real part of a complex function.*]

B6.a 1999

Fluid Dynamics

1. In the usual notation, the Navier-Stokes equations are

$$\sum_i \frac{\partial u_i}{\partial x_i} = 0, \quad \rho\frac{du_i}{dt} = -\frac{\partial p}{\partial x_i} + \sum_j \frac{\partial}{\partial x_j}\left(\mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)\right)$$

and

$$\rho c\frac{dT}{dt} = k\sum_i \frac{\partial^2 T}{\partial x_i^2} + \tfrac{1}{2}\sum_{i,j}\mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)^2,$$

where $\mu = \mu(T)$ is the viscosity. To which conservation laws do these equations correspond, and what thermodynamic assumption has been made?

Show that there is a solution in which $x_2 = y$, $u_1 = u(y)$, $T = T(y)$, $u_2 = u_3 = 0$ as long as $p =$ const. and

$$\frac{d}{dy}\left(\mu\frac{du}{dy}\right) = 0, \quad k\frac{d^2T}{dy^2} + \mu\left(\frac{du}{dy}\right)^2 = 0.$$

Can such a situation describe flow between two parallel plates, one of which is moved relative to the other under the action of a prescribed constant shear stress τ_w when (a) the plates are thermally insulated and (b) the plates are isothermal and $\mu = \mu_0 e^T$ where μ_0 is constant? [You may use the fact that $\int_0^{T_0} \frac{dT}{\sqrt{e^{T_0} - e^T}}$ increases from zero at $T_0 = 0$ to a maximum and then decreases as $T_0 \to \infty$.]

B6.a, 1998

Fluid Dynamics

1. The Navier-Stokes equations describing two-dimensional incompressible flow can be written in the form
$$\rho\left[\frac{\partial(\nabla^2\psi)}{\partial t}+\frac{\partial(\psi,\nabla^2\psi)}{\partial(y,x)}\right]=\mu\nabla^4\psi,$$
where ψ is the stream function. When the characteristic velocity and length are U and L respectively, show that the dimensionless stream function satisfies
$$\frac{\partial(\nabla^2\psi)}{\partial t}+\frac{\partial(\psi,\nabla^2\psi)}{\partial(y,x)}=\frac{1}{Re}\nabla^4\psi, \qquad (*)$$
where $Re=\dfrac{\rho UL}{\mu}$ and ψ, x, y and t are now dimensionless.

Fluid flows steadily in a two-dimensional wedge defined by $-\alpha<\theta<\alpha$ in polar coordinates under the action of a source at the origin whose volume flow rate (per unit width transverse to the flow) is M. Show that a suitable scaling for the velocity at distance L from the source is $M/2\alpha L$ and that $Re = M\rho/2\alpha\mu$. Show that a possible flow is $\psi=f(\theta)$ where
$$f''''+4f''+2Ref'f''=0$$
with $f'(\pm\alpha)=0$, $f(\alpha)-f(-\alpha)=2\alpha$.

Find an approximate solution for $v=f'(\theta)$ when $Re\ll 1$, and sketch the graph of $v(\theta)$. How do you expect the profile for v to be modified if $Re\gg 1$?

[*In plane polar coordinates* $\nabla^2=\dfrac{\partial^2}{\partial r^2}+\dfrac{1}{r}\dfrac{\partial}{\partial r}+\dfrac{1}{r^2}\dfrac{\partial^2}{\partial\theta^2}$.]

B6.a 1997

Fluid Dynamics

1. Show that if a liquid has velocity $\mathbf{u}$ and if $V(t)$ is a volume that always contains the same fluid particles, then

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{V(t)} F\mathrm{d}V = \int_{V(t)} \{\frac{\partial F}{\partial t} + \mathbf{u}.\nabla F + F(\nabla.\mathbf{u})\}\mathrm{d}V,$$

for any differentiable function F.

Given that the stress tensor in an incompressible Newtonian fluid is

$$\tau_{ij} = -p\delta_{ij} + \mu(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}),$$

derive the Navier-Stokes equations,

$$\nabla.\mathbf{u} = 0, \quad \rho\frac{\mathrm{d}\mathbf{u}}{\mathrm{d}t} = -\nabla p + \mu\nabla^2\mathbf{u},$$

in the usual notation. Show further that the principle of conservation of energy can be written as

$$\rho c\frac{\mathrm{d}T}{\mathrm{d}t} = k\nabla^2 T + \Phi,$$

where c is the specific heat, k the constant thermal conductivity and

$$\Phi = \frac{1}{2}\mu\sum_{i,j}(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i})^2.$$

Deduce that, in any viscous flow in an insulated container, mechanical energy is always converted into thermal energy.