1. PS7, Q1: Bessel's equation and Bessel functions

   Consider Bessel's differential equation of order n:

$$x^2y'' + xy' + (x^2 - n^2)y = 0,$$

   for an integer $n \geq 0$.

   (a) Explain how do you determine whether a point $x = a$ of a linear second-order differential equation is analytic or singular. What does it mean to be regular singular versus irregular singular? Classify the point $x = 0$ for this equation.

   (b) Find the indicial exponents, α_1, α_2 with $\Re(\alpha_1) > \Re(\alpha_2)$ for a local (Frobenius) series expansions at $x = 0$. Determine the expansion $y = \sum_k a_k x^{k+\alpha_1}$, giving the a_k coefficients in closed form.

   (c) How must a_0 be chosen so that the final series is the expansion of the Bessel functions of the first kind,

$$J_n(x) = \left(\frac{x}{2}\right)^n \sum_{k=0}^{\infty} \frac{(-x^2/4)^k}{k!(k+n)!}$$

   (d) What are appropriate forms for the series expansion for a second, linearly independent solution in the case $n = 0$ and $n > 0$? Give the most specific answer you can give in each case without actually calculating coefficients.

2. Exam 2008, Q2

   Consider the following eigenvalue problem for $y(x)$:

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + y = -\lambda y, \quad 0 \leq x \leq 1, \quad y(0) = 0, \quad y'(1) = 0. \tag{1}$$

   (a) Calculate the adjoint operator and adjoint boundary conditions. Show that the problem is self-adjoint. What can you conclude about: (i) the eigenvalues? (ii) the inner products of different eigenfunctions?

   (b) Find the eigenvalues and eigenfunctions.

   (c) Consider the inhomogeneous problem

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + y = f(x), \quad y(0) = c_0, \quad y'(1) = c_1, \tag{2}$$

   for $f(x)$ is given. Explain how you can deduce from part b) that the solution to this problem must be unique.

   (d) In equation (2), if $c_0 = 0$ and $c_1 = 0$, the solution can be written in terms of the Green's function $g(x,t)$ as

$$y(x) = \int_0^1 g(x,t)f(t)\,\mathrm{d}t. \tag{3}$$

   Write an expression for the Green's function $g(x,t)$ in terms of the eigenfunctions of (1).

   (e) By using the relation between (2) and (3), determine the solution $z(x)$ of the following integral equation without doing any integration:

$$h(x) = \int_0^1 g(x,t)z(t)\,\mathrm{d}t \quad \text{with } h(x) = x^2 - 2x.$$

   Note that $h(0) = 0$ and $h'(1) = 0$.

3. Exam 2008, Q3

   Consider the linear operator

$$Ly \equiv y(x) + \int_0^\infty [12e^{-2x-t} - 30e^{-x-2t}]y(t)\,\mathrm{d}t.$$

   (a) Determine the adjoint operator

   (b) Determine all the eigenvalues for $Ly = \lambda y$. Determine the eigenfunctions for the eigenvalues of finite multiplicity.

   (c) Determine the adjoint eigenfunctions for the eigenvalues of finite multiplicity.

   (d) Consider the inhomogeneous problem

$$Ly = ce^{-2x} + 6e^{-x}$$

   for all values of the parameter c. For which values of c is the solution unique? For which values of c is there no solution? For which values of c do multiple solutions exist?