Keywords for PS4: Sturm-Liouville form of ODE; determining eigenvalues and eigenfunctions of BVP; eigenfunction expansions for solutions of BVP

Keywords for PS5: applying FAT for BVPs; degenerate Fredholm integral operators and equations (FIE); determinine eigenvalues and eigenfunctions of FIE; applying FAT for FIE.

Keywords for PS6: analytic vs. singular points of ODE; regular singular vs. irregular singular points of ODE; indicial equations; form of series expansions for the three cases of indicial roots (two roots not separated by an integer; two equal roots; two roots separated by an integer).

1. PS4, Q1: Eigenfunction expansion for inhomogeneous problem
   (a) Find the general homogeneous solution of the Cauchy-Euler equation
   $$x^2y'' + 3xy' + (1 + \alpha y) = 0,$$
   where α is a given positive constant.
   (b) What does it mean to put a second-order linear differential equation into Sturm-Liouville form? Place the above equation into such a form.
   (c) Use a) to determine the eigenvalues and eigenfunctions of the Sturm-Liouville problem on $1 \le x \le e$
   $$\frac{\mathrm{d}}{\mathrm{d}x}\left(x^3\frac{\mathrm{d}y}{\mathrm{d}x}\right) + \lambda xy = 0, \qquad y(1) = 0,\ y(e) = 0.$$
   (d) Use b) to obtain the eigenfunction expanion for the solution of the inhomogeneous problem
   $$\frac{\mathrm{d}}{\mathrm{d}x}\left(x^3\frac{\mathrm{d}y}{\mathrm{d}x}\right) = x, \qquad y(1) = 0,\ y(e) = 0.$$

2. PS5 Q1: FAT for inhomogeneous BVPs

   Use the Fredholm alternative theorem to determine the parameter values (A, B) that yield existence of a solution for each of the BVPs:
   (a) For $0 \le x \le 2\pi$:
   $$y'' + y = A\sin x + B\cos x + 2\sin(x + \pi/3) + \sin^3 x, \qquad y(0) = 2(2\pi), \quad y'(0) = y'(2\pi)$$
   (b) On $0 \le x \le 1$:
   $$y'' + 2y' + y = 1, \qquad y'(0) + y(0) = A, \quad y'(1) = y(1) = 3.$$

3. PS5, Q2d

   Consider the FIE:
   $$\mathcal{L}y = \frac{6}{5\pi} + \int_0^1 \sin(2\pi x - 3\pi t)\, y(t)\, \mathrm{d}t \tag{1}$$
   (a) Determine all eigenfunctions and eigenvalues of finite multiplicity. State the eigenvalue with infinite multiplicity λ_∞?
   (b) Solve the inhomogeneous equation. Are solutions unique? If not, then find the simplest homogeneous solution that may be added to the particular solution.
   - $f(x) = 2\sin(2\pi x - \pi/6)$
   - $f(x) = \sin(2\pi x)$
   - General forcing, $f(x)$

4. PS6: Summary

   Write a summary of all important cases of the Frobenius theorem for series expansions and the resultant series forms (in the cases where the roots of the indicial equation are not separated by integer, are equal, or are separated by an integer).

5. PS6, Q3: On a singular point at infinity Consider the differential equation

$$x^3 y'' + y = 0. \tag{5}$$

   (a) Use the transformation $x = 1/t$ to show the ODE has a regular singular point at $x = \infty$ and determine its indicial exponents.

   (b) Show that one solution to the ODE has a Taylor series at $x = \infty$. Find the series and assuming $y(\infty) = 1$, find $y(1)$ correct to three decimal places.

   (c) The leading behaviour of a particular solution of (5) is $y \sim x$ as $x \to \infty$. What is the next largest term in the expansion?