B5a

Techniques in applied mathematics

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1. **A review of the theory**

(a) Write down the most general n^{th} order linear ordinary differential equation (with variable coefficients) given by

$$\mathcal{L}[y] = f(x), \tag{1a}$$

for an operator $\mathcal{L}$ that you should define. You may assume that the ODE is to be solved on $a \leq x \leq b$, and the boundary conditions are of the form

$$D_i[y] = d_i, \qquad i = 1, 2, 3, \ldots, n \tag{1b}$$

for a constant d_i, and for an operator D_i that involves linear combinations of y and its first $n-1$ derivatives at the points $x = a, b$.

(b) Describe the process of using a Green's function, $G(x, \xi)$, to solve the BVP (1). Be sure to specify exactly how the Green's function is defined, and then how it can be solved. What are the conditions on G at the point $x = \xi$?

Solution:

(a) Define:

$$\mathcal{L}_x[y] = a_n(x)y^{(n)} + a_{n-1}(x)y^{(n-1)} + \ldots\ldots a_1(x)y' + a_0(x)y = f(x).$$

(note that subscript x is to remind us of the variable of differentiation)

(b)
1. GF should satisfy:

$$\mathcal{L}_x[G] = \delta(x - \xi) \qquad \text{for } a < x, \xi < b$$

and homogeneous boundary conditions $D_i[G] = 0$.

2. We split the GF into two parts: a left part ($x < \xi$) and a right part ($x > \xi$). The ODE then provides us with two general solutions:

$$G_L(x, \xi) = \sum_{i=1}^{n} c_i y_i \quad \text{and} \quad G_R(x, \xi) = \sum_{i=1}^{n} d_i y_i,$$

and hence $2n$ unknown coefficients. The boundary conditions on either side of the BVP provide n conditions. The other n conditions are at $x = \xi$.

3. At $x = \xi$, the dominant balance in the GF is due to the large derivatives and singularity:

$$a_n(\xi)G^{(n)}(x,\xi) \sim \delta(x-\xi),$$

where it is assumed that the coefficients $a_i(x)$ are smooth at $x = \xi$. Integrating once gives

$$G^{(n-1)}(x,\xi) \sim \frac{H(x-\xi)}{a_n(\xi)} + C(x).$$

Note that because $H(x-\xi)$ possesses a jump at $x = \xi$, then $C(x)$ may only possess a jump (rather than blowup) since it must be less singular. We can continue integrating, but any of the lower derivatives will be continuous at $x = \xi$ (why?). This gives us the n necessary conditions:

$$G, G', G'', \ldots, G^{(n-2)} \text{ are continuous at } x = \xi \tag{2}$$

$$\left[G^{(n)}\right]_{x=\xi^-}^{x=\xi^+} = \frac{1}{a_n(\xi)}. \tag{3}$$

2. **Finding Green's functions**

Obtain the Green's function to

$$(x+3)y''' = f(x), \qquad 0 \le x \le 1$$
$$y(0) = 0,\ y'(0) = 0 \quad \text{and} \quad y'(1) = \frac{1}{2}.$$

Provide a sketch of the Green's function.

Solution: Green's function:

$$g(x,\xi) = \begin{cases} \frac{\xi-1}{2}x^2 & x < \xi \\ \frac{\xi}{2}x^2 - \xi x + \frac{\xi^2}{2} & x > \xi \end{cases}$$

Homogeneous solution with inhomogeneous BVs is $y_s = \frac{1}{2}x^2$, so

$$y = \int_0^1 G(x,\xi)\frac{f(\xi)}{x+3}\,d\xi + \frac{1}{2}x^2.$$

3. **Definitions**: Define the following terms:
   (a) Functional
   (b) Test functions and the class C_0^∞
   (c) Distribution
   (d) *Regular* distribution

Solution:

(a) A **functional** is a map from a vector space to a scalar field, i.e.

$$F : C_0^\infty(\mathbb{R}) \mapsto \mathbb{R}.$$

We can think of functionals as *acting* on functions and returning real numbers; this is in contrast to functions that act on numbers and return numbers.

(b) A function $\phi : \mathbb{R} \mapsto \mathbb{R}$ is a **test function** if (a) ϕ is infinitely differentiable and (b) ϕ has compact support, that is, $\phi(x) = 0$ for all $x \notin [-X, X]$ for some $X > 0$.

The set of all test functions (on $\mathbb{R}$ and centered about 0) is written $C_0^\infty(\mathbb{R})$.

(c) A **distribution** is a linear functional that is continuous. This means that:

1. $< u, \alpha\phi + \beta\psi >= \alpha < u, \phi > +\beta < u, \psi >$
2. For all $X > 0$, there exists $C > 0$ and $N \geq 0$ such that

$$| < u, \phi > | \leq C \sum_{k \leq N} \max \left| \frac{d^k \phi}{dx^k} \right|,$$

(d) A function can *induce* a functional. In particular, every locally integrable function defines a **regular distribution** through

$$< f, \phi >= \int_{-\infty}^{\infty} f(x)\phi(x)\, dx.$$

4. **Problem Set 2, Q1**

Show that

$$f_n(x) = \frac{e^{-nx^2/4}}{\sqrt{4\pi/n}}$$

converges to the δ distribution in the limit $n \to \infty$ using the notion of convergence of distributions.

Solution: Remember that we are concerned with *regular* distributions (that is, we get to use the integral formulation of the functional). The goal is to show that

$$I(n) =< f_n, \phi >= \int_{-\infty}^{\infty} f_n(x)\phi(x)\, dx \to \phi(0),$$

as $n \to \infty$ for $\phi \in C_0^\infty(\mathbb{R})$. We first split the integral into four pieces:

$$I(n) = \overbrace{\int_{-\infty}^{-p} f_n(x)[\phi(x) - \phi(0)]\, dx}^{I_1(n)} + \overbrace{\int_{-p}^{p} f_n(x)[\phi(x) - \phi(0)]\, dx}^{I_2(n)} + \underbrace{\int_{p}^{\infty} f_n(x)[\phi(x) - \phi(0)]\, dx}_{I_3(n)} + \underbrace{\int_{-\infty}^{\infty} f_n(x)\phi(0)\, dx}_{I_4(n)}.$$

Now the last integral

$$I_4(n) = \phi(0) \int_{-\infty}^{\infty} \frac{\sqrt{n}}{\sqrt{4\pi}} e^{-nx^2/4}\, dx = \phi(0).$$

It remains to show that $|I_1 + I_2 + I_3| \to 0$. First, look at

$$|I_2(n)| \leq \int_{-p}^{p} f_n(x)|\phi(x) - \phi(0)|\, dx.$$

The trick is to notice that if x is close to 0, then $\phi(x) \sim \phi(0)$ and this allows us to make the integral as small as we like. So for a given $\epsilon > 0$, we select a p with $|x - 0| < p$ so as to force $|\phi(x) - \phi(0)| < \epsilon/2$. Consequently,

$$|I_2(x)| \leq \frac{\epsilon}{2} \int_{-\infty}^{\infty} f_n(x)\, dx = \frac{\epsilon}{2}.$$

Now it remains to bound

$$|I_1(n) + I_3(n)| \leq \left(\int_{-\infty}^{-p} + \int_{p}^{\infty} \right) f_n(x)|\phi(x) - \phi(0)|\, dx.$$

What we *do* know is that $|\phi(x) - \phi(0)|$ must be bounded for all x since ϕ is continuous. Thus we may write $|\phi(x) - \phi(0)| \leq 2M$ for some $M > 0$. This leaves us trying to bound

$$|I_1(n) + I_3(n)| \leq 2M \left(\int_{-\infty}^{-p} + \int_{p}^{\infty} \right) f_n(x)\, dx.$$

Now remember that p is a *fixed* number (we fixed it in setting a bound on I_2). However, we expect that as $n \to \infty$, the tails of the function $f_n(x)$ thin out, so this quantity should be small as $n \to \infty$ (due to the exponential decay). You can see this because

$$e^{-nx^2/4} = e^{-s^2},$$

gives you a normalized function, but point $x = p$ becomes $s = \pm\sqrt{n}p/2$. In other words, once renormalized, the two integrals above are integrated only at ∞ for large n (giving an area of zero). Let us now put this into mathematics. It's easier to write

$$|I_1(n) + I_3(n)| \leq 2M \left(\int_{-\infty}^{\infty} - \int_{-p}^{p} \right) f_n(x)\, dx = 2M \left(1 - \int_{-p}^{p} f_n(x)\, dx \right).$$

Now

$$\int_{-p}^{p} f_n(x)\, dx = \frac{1}{\sqrt{\pi}} \int_{-\sqrt{n}p/2}^{\sqrt{n}p/2} e^{-s^2}\, ds \to 1$$

as $n \to \infty$. Thus we can select a $N > 0$ such that the above quantity can be made as close to 1 as we like. We shall then choose such N so that

$$|I_1(n) + I_3(n)| \leq 2M \left(1 - \int_{-p}^{p} f_n(x)\, dx \right) \leq 2M \frac{\epsilon}{4M} = \frac{\epsilon}{2}.$$

We're done. We have thus guaranteed that given any $\epsilon > 0$, we can select an $N > 0$ such that for $n > N$.

$$|I(n) - \phi(0)| \leq |I1(n) + I_3(n)| + |I_2(n)| \leq \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

Thus

$$\lim_{n\to\infty} < f_n, \phi >= \phi(0),$$

so the limit of f_n must be the δ distribution.

5. **Problem Set 3, Q4**

Consider the eigenvalue problem on $0 \leq x \leq 1$,

$$y'' + 2y' + (1+\lambda)y = 0, \tag{4}$$
$$y'(0) + y(0) = 0, \tag{5}$$
$$y'(1) + y(1) = 0. \tag{6}$$

(a) Assuming λ to be a positive constant, what is the general solution of the homogeneous ODE. Apply the boundary conditions to determine the eigenvalues and eigenfunctions.

(b) What is the adjoint problem? Obtain the adjoint eigenfunctions.

(c) What are the functions p, q, r, that put this problem in standard Sturm-Liouville form[1]? Verify that the expected orthogonality conditions (i.e. the one in terms of y's and r and the one in terms of y's and w's) are satisfied by direct integration with the eigenfunctions.

(d) Use the eigenfunctions and their adjoints to obtain the coefficients in the eigenfunction expansion

$$y(x) = \sum_{k=0}^{\infty} c_k y_k$$

for the solution of the problem

$$y'' + 2y' + 2y = 1 \tag{7}$$
$$y'(0) + y(0) = 2 \tag{8}$$
$$y'(1) + y(1) = 3. \tag{9}$$

Solution:

(a) We solve $y'' + 2y' + (1 + \lambda y) = 0$ giving the general solution

$$y = Ae^{-x}\cos(\sqrt{\lambda}x) + Be^{-x}\sin(\sqrt{\lambda}x).$$

Then using

$$y'(0) + y(0) = B\sqrt{\lambda} = 0 \tag{10}$$
$$y'(1) + y(1) = -Ae^{-1}\sqrt{\lambda}\sin\sqrt{\lambda} = 0 \tag{11}$$

so we get $B = 0$ and the eigenvalues and functions,

$$y_k = e^{-x}\cos(\sqrt{\lambda_k}x) \qquad \lambda_k = (k\pi)^2, \qquad k = 0, 1, 2, \ldots$$

Note that when the solution of $y_0 = e^{-x}$ with $\lambda_0 = 0$ does hold.

[1] $\left(p(x)\frac{dy}{dx}\right)' + q(x)y = -\lambda r(x)y$

(b) The original problem, posed in eigenvalue form, is

$$\mathcal{L}y \equiv y'' + 2y' + y = -\lambda y$$

We can get the adjoint by integration by parts:

$$\langle w, \mathcal{L}y \rangle = (wy' + 2wy - w'y)\Big|_0^1 + \int_0^1 y\,[w'' - 2w' + w]\; dx,$$

so

$$\mathcal{L}^* w \equiv w'' - 2w' + w,$$

and problem is not formally self-adjoint. We need the boundary values for the adjoint problem. Setting the boundary terms of the integration by parts to be zero, and using $y'(1) == -y(1)$ and $y'(0) = -y(0)$, we get

$$y(1)[w(1) - w'(1)] - y(0)[w(0) - w'(0)] = 0,$$

which (in the context of the adjoint problem) must hold for all values of $y(1)$ and $y(0)$. These give us the boundary conditions, which, when combined with the adjoint DE, gives

$$\mathcal{L}^* w = -\lambda w \tag{12}$$

$$w(1) - w'(1) = 0 \qquad \text{and} \qquad w(0) - w'(0) = 0. \tag{13}$$

Again, we can solve for the eigenvalues and eigenfunctions of this problem. After some work, we get

$$w_k = e^x \cos(\sqrt{\lambda_k} x) \qquad \lambda_k = (k\pi)^2, \qquad k = 0, 1, 2, \ldots$$

(c) Let us place the ODE into Sturm-Liouiville form. Multiplying by e^{2x} and rearranging gives

$$(e^{2x} y')' + e^{2x} y = -\lambda e^{2x} y,$$

which is in Sturm-Liouiville form with $p = q = r = e^{2x}$. We recall that when placed in such form, the orthogonality condition of the eigenfunctions holds. Let us verify:

$$\langle y_j, r y_k \rangle = \int_0^1 \cos j\pi x \cos k\pi x \, dx = \frac{1}{2} \int_0^1 [\cos(k+j)\pi x + \cos(k-j)\pi x]\, dx$$

$$= -\frac{1}{2}\left[\frac{\sin(k+j)\pi x}{(k+j)\pi} + \frac{\sin(k-j)\pi x}{(k-j)\pi}\right] = \begin{cases} 0 & k \neq j \\ 1/2 & k = j \\ 1 & k = 1 \end{cases} \tag{14}$$

which verifies the orthogonality SL condition. However, we also know that we can also check the orthogonality of the adjoint eigenfunctions:

$$\langle y_j, w_k \rangle = \int_0^1 \cos(j\pi x) e^{-x} \cos(k\pi x) e^x \, dx.$$

which yields exactly the same answer as the previous integral computation.

(d) How do we obtain the coefficients of the eigenfunction expansion? Let us first place the ODE into the form

$$(y'' + 2y' + y) + y = \mathcal{L}y + y = 1.$$

Now we profit from the orthogonality with the adjoint eigenfunctions. Taking the inner product gives

$$\langle w_k, \mathcal{L}y\rangle + \langle w_k, y\rangle = \langle w_k, 1\rangle.$$

Using the boundary values of y, we get for first and last term

$$\begin{aligned}\langle w_k, \mathcal{L}y\rangle &= 3w_k(1) - 2w_k(0) - \lambda_k c_k \langle w_k, y_k\rangle \\ \langle w_k, 1\rangle &= \frac{(-1)^k e - 1}{1 + (k\pi)^2}\end{aligned}$$

We now use the fact that

$$\langle w_k, y_k\rangle = \int_0^1 \cos^2(k\pi x)\, dx = \begin{cases} \frac{1}{2} & k \neq 0 \\ 1 & k = 0 \end{cases} = \frac{d_k}{2}$$

where d_k is defined in the obvious way. Then using the boundary values of w_k, we get

$$3e(-1)^k - 2 + \frac{d_k}{2} c_k (1 - (k\pi)^2) = \frac{(-1)^k e - 1}{1 + (k\pi)^2}.$$

This allows us to solve for c_k

$$c_k = -\frac{1}{d_k} \frac{2}{1 - (k\pi)^2} \left[3e(-1)^k - 2 - \frac{(-1)^k e - 1}{1 + (k\pi)^2} \right],$$

where we set $d_k = 1$ for $k = 1, 2, \ldots$. For the case that $k = 0$, we have $d_k = 2$ and

$$c_0 = 2e + 1.$$