B5a
Techniques in applied mathematics

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1. A review of the theory

(a) Write down the most general n^{th} order linear ordinary differential equation (with variable coefficients) given by

$$\mathcal{L}[y] = f(x), \tag{1a}$$

for an operator $\mathcal{L}$ that you should define. You may assume that the ODE is to be solved on $a \leq x \leq b$, and the boundary conditions are of the form

$$D_i[y] = d_i, \qquad i = 1, 2, 3, \ldots, n \tag{1b}$$

for a constant d_i, and for an operator D_i that involves linear combinations of y and its first $n-1$ derivatives at the points $x = a, b$.

(b) Describe the process of using a Green's function, $G(x, \xi)$, to solve the BVP (1). Be sure to specify exactly how the Green's function is defined, and then how it can be solved. What are the conditions on G at the point $x = \xi$?

2. **Finding Green's functions**

Obtain the Green's function to

$$(x+3)y''' = f(x), \qquad 0 \leq x \leq 1$$
$$y(0) = 0,\ y'(0) = 0 \quad \text{and} \quad y'(1) = \frac{1}{2}.$$

Provide a sketch of the Green's function.

3. **Definitions**: Define the following terms:
   (a) Functional
   (b) Test functions and the class C_0^∞
   (c) Distribution
   (d) *Regular* distribution

4. **Problem Set 2, Q1**

   Show that

$$f_n(x) = \frac{e^{-nx^2/4}}{\sqrt{4\pi/n}}$$

   converges to the δ distribution in the limit $n \to \infty$ using the notion of convergence of distributions.

5. **Problem Set 3, Q4**

Consider the eigenvalue problem on $0 \leq x \leq 1$,

$$y'' + 2y' + (1+\lambda)y = 0, \tag{4}$$
$$y'(0) + y(0) = 0, \tag{5}$$
$$y'(1) + y(1) = 0. \tag{6}$$

(a) Assuming λ to be a positive constant, what is the general solution of the homogeneous ODE. Apply the boundary conditions to determine the eigenvalues and eigenfunctions.

(b) What is the adjoint problem? Obtain the adjoint eigenfunctions.

(c) What are the functions p, q, r, that put this problem in standard Sturm-Liouville form[1]? Verify that the expected orthogonality conditions (i.e. the one in terms of y's and r and the one in terms of y's and w's) are satisfied by direct integration with the eigenfunctions.

(d) Use the eigenfunctions and their adjoints to obtain the coefficients in the eigenfunction expansion

$$y(x) = \sum_{k=0}^{\infty} c_k y_k$$

for the solution of the problem

$$y'' + 2y' + 2y = 1 \tag{7}$$
$$y'(0) + y(0) = 2 \tag{8}$$
$$y'(1) + y(1) = 3. \tag{9}$$

[1] $\left(p(x)\frac{dy}{dx}\right)' + q(x)y = -\lambda r(x)y$