B5a

Techniques in applied mathematics

The idea behind Sturm Liouville theory is to describe the solutions to a BVP in terms of a basis (the set of eigenfunctions). In this class, you'll practice using the properties of eigenvalues and eigenfunctions in order to study such BVPs (in particular, you'll expand the solutions in terms of the set of eigenfunctions).

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1. **Q4 from PS3**

Consider the eigenvalue problem on $0 \le x \le 1$,

$$y'' + 2y' + (1+\lambda)y = 0, \tag{1}$$
$$y'(0) + y(0) = 0, \tag{2}$$
$$y'(1) + y(1) = 0. \tag{3}$$

(a) Assuming λ to be a positive constant, what is the general solution of the homogeneous ODE. Apply the boundary conditions to determine the eigenvalues and eigenfunctions.

(b) What is the adjoint problem? Obtain the adjoint eigenfunctions.

(c) What are the functions p, q, r, that put this problem in standard Sturm-Liouville form[1]? Verify that the expected orthogonality conditions (i.e. the one in terms of y's and r and the one in terms of y's and w's) are satisfied by direct integration with the eigenfunctions.

(d) Use the eigenfunctions and their adjoints to obtain the coefficients in the eigenfunction expansion

$$y(x) = \sum_{k=0}^{\infty} c_k y_k$$

for the solution of the problem

$$y'' + 2y' + 2y = 1 \tag{4}$$
$$y'(0) + y(0) = 2 \tag{5}$$
$$y'(1) + y(1) = 3. \tag{6}$$

Here's an integration result you can use:

$$\int_0^1 \cos(k\pi x)\cos(j\pi x)\,dx = \frac{1}{2}\int_0^1 [\cos(k+j)\pi x + \cos(k-j)\pi x]\,dx = \begin{cases} 0 & k \ne j \\ \frac{1}{2} & k = j, k > 0 \\ 1 & k = j = 0 \end{cases}$$

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[1] $\left(p(x)\frac{dy}{dx}\right)' + q(x)y = -\lambda r(x)y$

Solution:

(a) We solve $y'' + 2y' + (1 + \lambda y) = 0$ giving the general solution

$$y = Ae^{-x}\cos(\sqrt{\lambda}x) + Be^{-x}\sin(\sqrt{\lambda}x).$$

Then using

$$y'(0) + y(0) = B\sqrt{\lambda} = 0 \tag{7}$$
$$y'(1) + y(1) = -Ae^{-1}\sqrt{\lambda}\sin\sqrt{\lambda} = 0 \tag{8}$$

so we get $B = 0$ and the eigenvalues and functions,

$$y_k = e^{-x}\cos(\sqrt{\lambda_k}x) \qquad \lambda_k = (k\pi)^2, \qquad k = 0, 1, 2, \ldots$$

Note that when the solution of $y_0 = e^{-x}$ with $\lambda_0 = 0$ does hold.

(b) The original problem, posed in eigenvalue form, is

$$\mathcal{L}y \equiv y'' + 2y' + y = -\lambda y$$

We can get the adjoint by integration by parts:

$$\langle w, \mathcal{L}y\rangle = (wy' + 2wy - w'y)\Big|_0^1 + \int_0^1 y\left[w'' - 2w' + w\right]\, dx,$$

so

$$\mathcal{L}^* w \equiv w'' - 2w' + w,$$

and problem is not formally self-adjoint. We need the boundary values for the adjoint problem. Setting the boundary terms of the integration by parts to be zero, and using $y'(1) == -y(1)$ and $y'(0) = -y(0)$, we get

$$y(1)[w(1) - w'(1)] - y(0)[w(0) - w'(0)] = 0,$$

which (in the context of the adjoint problem) must hold for all values of $y(1)$ and $y(0)$. These give us the boundary conditions, which, when combined with the adjoint DE, gives

$$\mathcal{L}^* w = -\lambda w \tag{9}$$
$$w(1) - w'(1) = 0 \qquad \text{and} \qquad w(0) - w'(0) = 0. \tag{10}$$

Again, we can solve for the eigenvalues and eigenfunctions of this problem. After some work, we get

$$w_k = e^{x}\cos(\sqrt{\lambda_k}x) \qquad \lambda_k = (k\pi)^2, \qquad k = 0, 1, 2, \ldots$$

(c) Let us place the ODE into Sturm-Liouiville form. Multiplying by e^{2x} and rearranging gives

$$(e^{2x}y')' + e^{2x}y = -\lambda e^{2x}y,$$

which is in Sturm-Liouiville form with $p = q = r = e^{2x}$. We recall that when placed in such form, the orthogonality condition of the eigenfunctions holds. Let us verify:

$$\begin{aligned}\langle y_j, ry_k\rangle &= \int_0^1 \cos j\pi x \cos k\pi x\, dx = \frac{1}{2}\int_0^1 [\cos(k+j)\pi x + \cos(k-j)\pi x]\, dx \\ &= -\frac{1}{2}\left[\frac{\sin(k+j)\pi x}{(k+j)\pi} + \frac{\sin(k-j)\pi x}{(k-j)\pi}\right] = \begin{cases} 0 & k \neq j \\ 1/2 & k = j \\ 1 & k = 1 \end{cases} \qquad (11)\end{aligned}$$

which verifies the orthogonality SL condition. However, we also know that we can also check the orthogonality of the adjoint eigenfunctions:

$$\langle y_j, w_k\rangle = \int_0^1 \cos(j\pi x)e^{-x}\cos(k\pi x)e^{x}\, dx.$$

which yields exactly the same answer as the previous integral computation.

(d) How do we obtain the coefficients of the eigenfunction expansion? Let us first place the ODE into the form

$$(y'' + 2y' + y) + y = \mathcal{L}y + y = 1.$$

Now we profit from the orthogonality with the adjoint eigenfunctions. Taking the inner product gives

$$\langle w_k, \mathcal{L}y\rangle + \langle w_k, y\rangle = \langle w_k, 1\rangle.$$

Using the boundary values of y, we get for first and last term

$$\begin{aligned}\langle w_k, \mathcal{L}y\rangle &= 3w_k(1) - 2w_k(0) - \lambda_k c_k\langle w_k, y_k\rangle \\ \langle w_k, 1\rangle &= \frac{(-1)^k e - 1}{1 + (k\pi)^2}\end{aligned}$$

We now use the fact that

$$\langle w_k, y_k\rangle = \int_0^1 \cos^2(k\pi x)\, dx = \begin{cases} \frac{1}{2} & k \neq 0 \\ 1 & k = 0 \end{cases} = \frac{d_k}{2}$$

where d_k is defined in the obvious way. Then using the boundary values of w_k, we get

$$3e(-1)^k - 2 + \frac{d_k}{2}c_k(1 - (k\pi)^2) = \frac{(-1)^k e - 1}{1 + (k\pi)^2}.$$

This allows us to solve for c_k

$$c_k = -\frac{1}{d_k}\frac{2}{1-(k\pi)^2}\left[3e(-1)^k - 2 - \frac{(-1)^k e - 1}{1+(k\pi)^2}\right],$$

where we set $d_k = 1$ for $k = 1, 2, \ldots$. For the case that $k = 0$, we have $d_k = 2$ and

$$c_0 = 2e - 1.$$

2. **What is wrong with this argument?**

- $\mathcal{L}$ is a second-order differential operator with boundary conditions BC
- Eigenfunctions and eigenvalues are $\mathcal{L}y_k = -\lambda_k y_k$
- Eigenfunctions and eigenvalues of adjoint problem are $\mathcal{L}^* w_k = -\lambda_k w_k$

Using an eigenfunction expansion, $y = \sum c_k y_k$, the solution of the problem $\mathcal{L}y = f$ with BC $\neq 0$ is given by the following derivation:

$$\begin{aligned}
& & \mathcal{L}y &= f \\
&\Rightarrow & \mathcal{L}\sum_k c_k y_k &= f \\
&\Rightarrow & \sum_k c_k \mathcal{L} y_k &= f \\
&\Rightarrow & \sum_k -c_k \lambda_k y_k &= f \\
& & w_j \sum_k c_k \lambda_k y_k &= w_j f \\
&\Rightarrow & \sum_k -c_k \lambda_k \langle w_j, y_k \rangle &= \langle w_j, f \rangle
\end{aligned}$$

(a) Intuitively (in terms of the boundary conditions), why do you expect the above argument to be wrong?

(b) Some people named the problem as the differential operator interchanging with the summation on the second line. Is this problematic? Why or why not?

(c) The problem, in fact, lies with the very last line. By considering the eigenvalues and eigenfunctions of

$$\mathcal{L}y = y'', \qquad y(0) = 0, y(1) = 0,$$

show that proceeding from the second-to-last line and the last line is fallacious (in particular, you will need to argue that the functions under consideration are *not* uniformly continuous).

(d) Give the *current* derivation for the eigenfunction coefficients, c_k.

Solution:

(a) Valid for any boundary conditions—but we know BVP problems are quite sensitive to boundary conditions.

(b) This *can* be problematic. But for Sturm Liouville problems, the eigenfunctions $\{y_k\}$ form a complete set, meaning that any y with $\int y^2 r \, dx < \infty$ can be expanded into an eigenfunction series. The proof of this relies upon establishing the fact that

$$y = \sum_{k=0}^{\infty} c_k y_k$$

is uniformly convergent. Uniform convergence allows us to exchange derivatives with infinite sums.[2]

(c) Consider $\mathcal{L}y = y'' = -\lambda y$. The eigenfunctions are $w_k = y_k = \sin(k\pi x)$ and the eigenvalues are $\lambda_k = (k\pi)^2$. The problem is that we need to show

$$\int_0^1 \sum_{k=1}^{\infty} k^2 \sin(j\pi x)\sin(k\pi x)\,dx = \sum_{k=1}^{\infty} \int_0^1 k^2 \sin(j\pi x)\sin(k\pi x)\,dx.$$

Is it? Without loss of generality, set $j = 1$. Then notice that $\sin(\pi x)\sin(k\pi x)$ is a very rapidly oscillating wave (as $k \to \infty$) contained within an envelope given by the slowly varying amplitude $\sin(\pi x)$. In other words, for fixed x, the sequence $|k^2 \sin(\pi x)\sin(k\pi x)|$ is not bounded by any number as $k \to \infty$, and this is an obvious requirement for the partial sums of the series to be uniformly convergent.

(d)

$$\begin{aligned}
\langle w_j, \mathcal{L}y\rangle &= \langle w_j, f\rangle \\
\text{Boundary terms} + \langle \mathcal{L}^* w_j, y\rangle &= \langle w_j, f\rangle \\
-\lambda_j \int_0^1 w_j \sum_k c_k y_k\,dx &= \langle w_j, f\rangle \\
-\lambda_j \sum_k \int_0^1 c_k w_j y_k\,dx &= \langle w_j, f\rangle \\
-\lambda_j c_k \langle w_j, y_k\rangle &= \langle w_j, f\rangle
\end{aligned}$$

Note that it is quite crucial that we went the route of the adjoint operator, $\mathcal{L}^*$ as this allows us to work with the sum of $c_k y_k$ rather than $\lambda_k c_k y_k$.

3. Let $\mathcal{L}y = -\lambda y$ be an eigenvalue problem where $\mathcal{L}$ is a second-order linear differential operator. Prove or disprove the following statement: if the ODE can be manipulated into Sturm-Liouville form, then $\mathcal{L}$ is self-adjoint.

Solution: False. Try $\mathcal{L}y = y'' - y$. Then we can write $(e^{-x}y')' = -\lambda e^{-x}y$, but $\mathcal{L}^*w = w'' + w$ and the problem is not self-adjoint.